
Distributed Power-law Graph Computing: Theoretical and Empirical Analysis

Cong Xie

Dept. of Comp. Sci. and Eng.
Shanghai Jiao Tong University
800 Dongchuan Road
Shanghai 200240, China
xcgoner1108@gmail.com

Ling Yan

Dept. of Comp. Sci. and Eng.
Shanghai Jiao Tong University
800 Dongchuan Road
Shanghai 200240, China
yling0718@sjtu.edu.cn

Wu-Jun Li

National Key Lab. for Novel Software Tech.
Dept. of Comp. Sci. and Tech.
Nanjing University
Nanjing 210023, China
liwujun@nju.edu.cn

Zhihua Zhang

Dept. of Comp. Sci. and Eng.
Shanghai Jiao Tong University
800 Dongchuan Road
Shanghai 200240, China
zhang-zh@cs.sjtu.edu.cn

Abstract

With the emergence of big graphs in a variety of real applications like social networks, machine learning based on distributed graph-computing (DGC) frameworks has attracted much attention from big data machine learning community. In DGC frameworks, the graph partitioning (GP) strategy plays a key role to affect the performance, including the workload balance and communication cost. Typically, the degree distributions of natural graphs from real applications follow skewed power laws, which makes GP a challenging task. Recently, many methods have been proposed to solve the GP problem. However, the existing GP methods cannot achieve satisfactory performance for applications with power-law graphs. In this paper, we propose a novel vertex-cut method, called *degree-based hashing* (DBH), for GP. DBH makes effective use of the skewed degree distributions for GP. We theoretically prove that DBH can achieve lower communication cost than existing methods and can simultaneously guarantee good workload balance. Furthermore, empirical results on several large power-law graphs also show that DBH can outperform the state of the art.

1 Introduction

Recent years have witnessed the emergence of big graphs in a large variety of real applications, such as the web and social network services. Furthermore, many machine learning and data mining algorithms can also be modeled with graphs [14]. Hence, machine learning based on distributed graph-computing (DGC) frameworks has attracted much attention from big data machine learning community [14, 16, 15, 7, 12, 8]. To perform distributed (parallel) graph-computing on clusters with several machines (servers), one has to partition the whole graph across the machines in a cluster. Graph partitioning (GP) can dramatically affect the performance of DGC frameworks in terms of workload balance and communication cost. Hence, the GP strategy typically plays a key role in DGC frameworks. The ideal GP method should minimize the cross-machine communication cost, and simultaneously keep the workload in every machine approximately balanced.

Existing GP methods can be divided into two main categories: edge-cut and vertex-cut methods. Edge-cut tries to evenly assign the vertices to machines by cutting the edges. In contrast, vertex-cut tries to evenly assign the edges to machines by cutting the vertices. Figure 1 illustrates the edge-cut and vertex-cut partitioning results of an example graph. In Figure 1 (a), the edges (A,C) and (A,E) are cut, and the two machines store the vertex sets $\{A,B,D\}$ and $\{C,E\}$, respectively. In Figure 1 (b), the vertex A is cut, and the two machines store the edge sets $\{(A,B), (A,D), (B,D)\}$ and $\{(A,C), (A,E), (C,E)\}$, respectively. In edge-cut, both machines of a cut edge should maintain a ghost (local replica) of the vertex and the edge data. In vertex-cut, all the machines associated with a cut vertex should maintain a mirror (local replica) of the vertex. The ghosts and mirrors are shown in shaded vertices in Figure 1. In edge-cut, the workload of a machine is determined by the number of vertices located in that machine, and the communication cost of the whole graph is determined by the number of edges spanning different machines. In vertex-cut, the workload of a machine is determined by the number of edges located in that machine, and the communication cost of the whole graph is determined by the number of mirrors of the vertices.

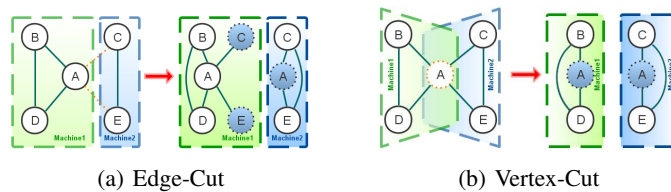


Figure 1: Two strategies for graph partitioning. Shaded vertices are ghosts and mirrors, respectively.

Most traditional DGC frameworks, such as GraphLab [14] and Pregel [16], use edge-cut methods [10, 19, 20, 21] for GP. Very recently, the authors of PowerGraph [7] find that the vertex-cut methods can achieve better performance than edge-cut methods, especially for power-law graphs. Hence, vertex-cut has attracted more and more attention from DGC research community. For example, PowerGraph [7] adopts a *random* vertex-cut method and two greedy variants for GP. GraphBuilder [9] provides some heuristics, such as the *grid*-based constrained solution, to improve the random vertex-cut method.

Large natural graphs usually follow skewed degree distributions like power-law distributions, which makes GP challenging. Different vertex-cut methods can result in different performance for power-law graphs. For example, Figure 2 (a) shows a toy power-law graph with only one vertex having much higher degree than the others. Figure 2 (b) shows a partitioning strategy by cutting the vertices $\{E, F, A, C, D\}$, and Figure 2 (c) shows a partitioning strategy by cutting the vertices $\{A, E\}$. We can find that the partitioning strategy in Figure 2 (c) is better than that in Figure 2 (b) because the number of mirrors in Figure 2 (c) is smaller which means less communication cost. The intuition underlying this example is that cutting higher-degree vertices can result in fewer mirror vertices. Hence, the power-law degree distribution can be used to facilitate GP. Unfortunately, existing vertex-cut methods, including those in PowerGraph and GraphBuilder, make rarely use of the power-law degree distribution for GP. Hence, they cannot achieve satisfactory performance in natural power-law graphs. PowerLyra [4] tries to combine both edge-cut and vertex-cut together by using the power-law degree distribution. However, it is lack of theoretical guarantee.

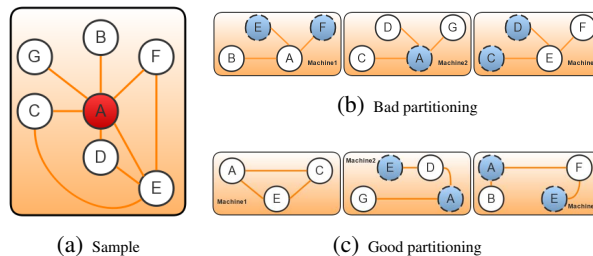


Figure 2: Partition a sample graph with vertex-cut.

In this paper, we propose a novel vertex-cut GP method, called *degree-based hashing* (DBH), for distributed power-law graph computing. The main contributions of DBH are briefly outlined as follows:

- DBH can effectively exploit the power-law degree distributions in natural graphs for vertex-cut GP.
- Theoretical bounds on the communication cost and workload balance for DBH can be derived, which show that DBH can achieve lower communication cost than existing methods and can simultaneously guarantee good workload balance.
- DBH can be implemented as an execution engine for PowerGraph [7], and hence all PowerGraph applications can be seamlessly supported by DBH.
- Empirical results on several large real graphs and synthetic graphs show that DBH can outperform the state-of-the-art methods.

2 Problem Formulation

Let $G = (V, E)$ denote a graph, where $V = \{v_1, v_2, \dots, v_n\}$ is the set of vertices and $E \subseteq V \times V$ is the set of edges in G . Let $|V|$ denote the cardinality of the set V , and hence $|V| = n$. v_i and v_j are called neighbors if $(v_i, v_j) \in E$. The degree of v_i is denoted as d_i , which measures the number of neighbors of v_i . Please note that we only need to consider the GP task for undirected graphs because the workload mainly depends on the number of edges no matter directed or undirected graphs the computation is based on. Even if the computation is based on directed graphs, we can also use the undirected counterparts of the directed graphs to get the partitioning results.

Assume we have a cluster of p machines. Vertex-cut GP is to assign each edge with the two corresponding vertices to one of the p machines in the cluster. The assignment of an edge is unique, while vertices may have replicas across different machines. For DGC frameworks based on vertex-cut GP, the workload (amount of computation) of a machine is roughly linear in the number of edges located in that machine, and the replicas of the vertices incur communication for synchronization. So the goal of vertex-cut GP is to *minimize* the number of replicas and simultaneously *balance* the number of edges on each machine.

Let $M(e) \in \{1, \dots, p\}$ be the machine edge $e \in E$ is assigned to, and $A(v) \subseteq \{1, \dots, p\}$ be the span of vertex v over different machines. Hence, $|A(v)|$ is the number of replicas of v among different machines. Similar to PowerGraph [7], one of the replicas of a vertex is chosen as the *master* and the others are treated as the *mirrors* of the master. We let $Master(v)$ denote the machine in which the master of v is located. Hence, the goal of vertex-cut GP can be formulated as follows:

$$\min_A \frac{1}{n} \sum_{i=1}^n |A(v_i)|$$

$$s.t. \max_m |\{e \in E \mid M(e) = m\}| < \lambda \frac{|E|}{p}, \text{ and } \max_m |\{v \in V \mid Master(v) = m\}| < \rho \frac{n}{p},$$

where $m \in \{1, \dots, p\}$ denotes a machine, $\lambda \geq 1$ and $\rho \geq 1$ are imbalance factors. We define $\frac{1}{n} \sum_{i=1}^n |A(v_i)|$ as *replication factor*, $\frac{p}{|E|} \max_m |\{e \in E \mid M(e) = m\}|$ as *edge-imbalance*, and $\frac{p}{n} \max_m |\{v \in V \mid Master(v) = m\}|$ as *vertex-imbalance*. To get a good balance of workload, λ and ρ should be as small as possible.

The degrees of natural graphs usually follow skewed power-law distributions [3, 1]:

$$\Pr(d) \propto d^{-\alpha},$$

where $\Pr(d)$ is the probability that a vertex has degree d and the power parameter α is a positive constant. The lower the α is, the more skewed a graph will be. This power-law degree distribution makes GP challenging [7]. Although vertex-cut methods can achieve better performance than edge-cut methods for power-law graphs [7], existing vertex-cut methods, such as *random* method in PowerGraph and *grid*-based method in GraphBuilder [9], cannot make effective use of the power-law distribution to achieve satisfactory performance.

3 Degree-Based Hashing for GP

In this section, we propose a novel vertex-cut method, called *degree-based hashing* (DBH), to effectively exploit the power-law distribution for GP.

3.1 Hashing Model

We refer to a certain machine by its index idx , and the idx th machine is denoted as P_{idx} . We first define two kinds of hash functions: vertex-hash function $idx = vertex_hash(v)$ which hashes vertex v to the machine P_{idx} , and edge-hash function $idx = edge_hash(e)$ or $idx = edge_hash(v_i, v_j)$ which hashes edge $e = (v_i, v_j)$ to the machine P_{idx} .

Our *hashing model* includes two main components:

- *Master-vertex assignment*: The *master* replica of v_i is uniquely assigned to one of the p machines with equal probability for each machine by some *randomized* hash function $vertex_hash(v_i)$.
- *Edge assignment*: Each edge $e = (v_i, v_j)$ is assigned to one of the p machines by some hash function $edge_hash(v_i, v_j)$.

It is easy to find that the above hashing model is a vertex-cut GP method. The *master-vertex assignment* can be easily implemented, which can also be expected to achieve a low vertex-imbalance score. On the contrary, the *edge assignment* is much more complicated. Different edge-hash functions can achieve different *replication factors* and different *edge-imbalance* scores. Please note that replication factor reflects communication cost, and edge-imbalance reflects workload-imbalance. Hence, the key of our hashing model lies in the edge-hash function $edge_hash(v_i, v_j)$.

3.2 Degree-Based Hashing

From the example in Figure 2, we observe that in power-law graphs the replication factor, which is defined as the total number of replicas divided by the total number of vertices, will be smaller if we cut vertices with relatively higher degrees. Based on this intuition, we define the $edge_hash(v_i, v_j)$ as follows:

$$edge_hash(v_i, v_j) = \begin{cases} vertex_hash(v_i) & \text{if } d_i < d_j, \\ vertex_hash(v_j) & \text{otherwise.} \end{cases} \quad (1)$$

It means that we use the vertex-hash function to define the edge-hash function. Furthermore, the edge-hash function value of an edge is determined by the degrees of the two associated vertices. More specifically, the edge-hash function value of an edge is defined by the vertex-hash function value of the associated vertex with a smaller degree. Hence, our method is called *degree-based hashing* (DBH). DBH can effectively capture the intuition that cutting vertices with higher degrees will get better performance.

Our DBH method for vertex-cut GP is briefly summarized in Algorithm 1, where $[n] = \{1, \dots, n\}$.

Algorithm 1 Degree-based hashing (DBH) for vertex-cut GP

Input: The set of edges E ; the set of vertices V ; the number of machines p .

Output: The assignment $M(e) \in [p]$ for each edge e .

- 1: Initialization: count the degree d_i for each $i \in [n]$ in parallel
 - 2: **for all** $e = (v_i, v_j) \in E$ **do**
 - 3: Hash each edge in parallel:
 - 4: **if** $d_i < d_j$ **then**
 - 5: $M(e) \leftarrow vertex_hash(v_i)$
 - 6: **else**
 - 7: $M(e) \leftarrow vertex_hash(v_j)$
 - 8: **end if**
 - 9: **end for**
-

4 Theoretical Analysis

In this section, we present theoretical analysis for our DBH method. For comparison, the *random* vertex-cut method (called *Random*) of PowerGraph [7] and the *grid*-based constrained solution (called *Grid*) of GraphBuilder [9] are adopted as baselines. Our analysis is based on randomization. Moreover, we assume that the graph is undirected and there are no duplicated edges in the graph. We mainly study the performance in terms of *replication factor*, *edge-imbalance* and *vertex-imbalance* defined in Section 2. Due to space limitation, we put the proofs of all theoretical results into the supplementary material.

4.1 Partitioning Degree-fixed Graphs

Firstly, we assume that the degree sequence $\{d_i\}_{i=1}^n$ is fixed. Then we can get the following expected replication factor produced by different methods.

Random assigns each edge evenly to the p machines via a randomized hash function. The result can be directly got from PowerGraph [7].

Lemma 1. *Assume that we have a sequence of n vertices $\{v_i\}_{i=1}^n$ and the corresponding degree sequence $D = \{d_i\}_{i=1}^n$. A simple randomized vertex-cut on p machines has the expected replication factor:*

$$\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n |A(v_i)| \middle| D \right] = \frac{p}{n} \sum_{i=1}^n \left[1 - \left(1 - \frac{1}{p} \right)^{d_i} \right].$$

By using the *Grid* hash function, each vertex has \sqrt{p} rather than p candidate machines compared to *Random*. Thus we simply replace p with \sqrt{p} to get the following corollary.

Corollary 1. *By using *Grid* for hashing, the expected replication factor on p machines is:*

$$\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n |A(v_i)| \middle| D \right] = \frac{\sqrt{p}}{n} \sum_{i=1}^n \left[1 - \left(1 - \frac{1}{\sqrt{p}} \right)^{d_i} \right].$$

Using DBH method in Section 3.2, we obtain the following result by fixing the sequence $\{h_i\}_{i=1}^n$, where h_i is defined as the number of v_i 's adjacent edges which are hashed by the neighbors of v_i according to the edge-hash function defined in (1).

Theorem 1. *Assume that we have a sequence of n vertices $\{v_i\}_{i=1}^n$ and the corresponding degree sequence $D = \{d_i\}_{i=1}^n$. For each v_i , $d_i - h_i$ adjacent edges of it are hashed by v_i itself. Define $H = \{h_i\}_{i=1}^n$. Our DBH method on p machines has the expected replication factor:*

$$\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n |A(v_i)| \middle| H, D \right] = \frac{p}{n} \sum_{i=1}^n \left[1 - \left(1 - \frac{1}{p} \right)^{h_i+1} \right] \leq \frac{p}{n} \sum_{i=1}^n \left[1 - \left(1 - \frac{1}{p} \right)^{d_i} \right],$$

where $h_i \leq d_i - 1$ for any v_i .

This theorem says that our DBH method has smaller expected replication factor than *Random* of PowerGraph [7].

Next we turn to the analysis of the balance constraints. We still fix the degree sequence and have the following result for our DBH method.

Theorem 2. *Our DBH method on p machines with the sequences $\{v_i\}_{i=1}^n$, $\{d_i\}_{i=1}^n$ and $\{h_i\}_{i=1}^n$ defined in Theorem 1 has the edge-imbalance:*

$$\frac{\max_m |\{e \in E \mid M(e) = m\}|}{|E|/p} = \frac{\sum_{i=1}^n \frac{h_i}{p} + \max_{j \in [p]} \sum_{v_i \in P_j} (d_i - h_i)}{2|E|/p}.$$

Although the master vertices are evenly assigned to each machine, we want to show how the randomized assignment is close to the perfect balance. This problem is well studied in the model of uniformly throwing n balls into p bins when $n \gg p(\ln p)^3$ [18].

Lemma 2. *The maximum number of master vertices for each machine is bounded as follows:*

$$\begin{cases} \Pr[\text{MaxLoad} > k_a] = o(1) & \text{if } a > 1, \\ \Pr[\text{MaxLoad} > k_a] = 1 - o(1) & \text{if } 0 < a < 1. \end{cases}$$

Here $\text{MaxLoad} = \max_m |\{v \in V \mid \text{Master}(v) = m\}|$, and $k_a = \frac{n}{p} + \sqrt{\frac{2n \ln p}{p} \left(1 - \frac{\ln \ln p}{2a \ln p}\right)}$.

4.2 Partitioning Power-law Graphs

Now we change the sequence of fixed degrees into a sequence of random samples generated from the power-law distribution. As a result, upper-bounds can be provided for the above three methods, which are *Random*, *Grid* and *DBH*.

Theorem 3. *Let the minimal degree be d_{\min} and each $d \in \{d_i\}_{i=1}^n$ be sampled from a power-law degree distribution with parameter $\alpha \in (2, 3)$. The expected replication factor of *Random* on p machines can be approximately bounded by:*

$$\mathbb{E}_D \left[\frac{p}{n} \sum_{i=1}^n \left(1 - \left(1 - \frac{1}{p}\right)^{d_i}\right) \right] \leq p \left[1 - \left(1 - \frac{1}{p}\right)^{\hat{\Omega}}\right],$$

where $\hat{\Omega} = d_{\min} \times \frac{\alpha-1}{\alpha-2}$.

This theorem says that when the degree sequence is under power-law distribution, the upper bound of the expected replication factor increases as α decreases. This implies that *Random* yields a worse partitioning when the power-law graph is more skewed.

Like Corollary 1, we replace p with \sqrt{p} to get the similar result for *Grid*.

Corollary 2. *By using *Grid* method, the expected replication factor on p machines can be approximately bounded by:*

$$\mathbb{E}_D \left[\frac{\sqrt{p}}{n} \sum_{i=1}^n \left(1 - \left(1 - \frac{1}{\sqrt{p}}\right)^{d_i}\right) \right] \leq \sqrt{p} \left[1 - \left(1 - \frac{1}{\sqrt{p}}\right)^{\hat{\Omega}}\right],$$

where $\hat{\Omega} = d_{\min} \times \frac{\alpha-1}{\alpha-2}$.

Note that $\sqrt{p} \left[1 - \left(1 - \frac{1}{\sqrt{p}}\right)^{\hat{\Omega}}\right] \leq p \left[1 - \left(1 - \frac{1}{p}\right)^{\hat{\Omega}}\right]$. So Corollary 2 tells us that *Grid* can reduce the replication factor but it is not motivated by the skewness of the degree distribution.

Theorem 4. *Assume each edge is hashed by our *DBH* method and $h_i \leq d_i - 1$ for any v_i . The expected replication factor of *DBH* on p machines can be approximately bounded by:*

$$\mathbb{E}_{H,D} \left[\frac{p}{n} \sum_{i=1}^n \left(1 - \left(1 - \frac{1}{p}\right)^{h_i+1}\right) \right] \leq p \left[1 - \left(1 - \frac{1}{p}\right)^{\hat{\Omega}'}\right],$$

where $\hat{\Omega}' = d_{\min} \times \frac{\alpha-1}{\alpha-2} - d_{\min} \times \frac{\alpha-1}{2\alpha-3} + \frac{1}{2}$.

Note that

$$p \left[1 - \left(1 - \frac{1}{p}\right)^{\hat{\Omega}'}\right] < p \left[1 - \left(1 - \frac{1}{p}\right)^{\hat{\Omega}}\right].$$

Therefore, our *DBH* method can expectedly reduce the replication factor. The term $\frac{\alpha-1}{2\alpha-3}$ increases as α decreases, which means our *DBH* reduces more replication factor when the power-law graph is more skewed. Note that *Grid* and our *DBH* method actually use two different ways to reduce the replication factor. *Grid* reduces more replication factor when p grows. These two approaches can be combined to obtain further improvement, which is not the focus of this paper.

Finally, we show that our *DBH* method also guarantees good edge-balance (workload balance) under power-law distributions.

Theorem 5. Assume each edge is hashed by the DBH method with d_{min} , $\{v_i\}_{i=1}^n$, $\{d_i\}_{i=1}^n$ and $\{h_i\}_{i=1}^n$ defined above. The vertices are evenly assigned. By taking the constant $2|E|/p = \mathbb{E}_D \left[\sum_{i=1}^n d_i \right] = n\mathbb{E}_D [d]/p$, there exists $\epsilon \in (0, 1)$ such that the expected edge-imbalance of DBH on p machines can be bounded w.h.p (with high probability). That is,

$$\mathbb{E}_{H,D} \left[\sum_{i=1}^n \frac{h_i}{p} + \max_{j \in [p]} \sum_{v_i \in P_j} (d_i - h_i) \right] \leq (1 + \epsilon) \frac{2|E|}{p}.$$

Note that any ϵ that satisfies $1/\epsilon \ll n/p$ could work for this theorem, which results in a tighter bound for large n . Therefore, together with Theorem 4, this theorem shows that our DBH method can reduce the replication factor and simultaneously guarantee good workload balance.

5 Empirical Evaluation

In this section, empirical evaluation on real and synthetic graphs is used to verify the effectiveness of our DBH method. The cluster for experiment contains 64 machines connected via 1 GB Ethernet. Each machine has 24 Intel Xeon cores and 96GB of RAM.

5.1 Datasets

The graph datasets used in our experiments include both synthetic and real-world power-law graphs. Each synthetic power-law graph is generated by a combination of two synthetic directed graphs. The in-degree and out-degree of the two directed graphs are sampled from the power-law degree distributions with different power parameters α and β , respectively. Such a collection of synthetic graphs is separated into two subsets: one subset with parameter $\alpha \geq \beta$ which is shown in Table 1(a), and the other subset with parameter $\alpha < \beta$ which is shown in Table 1(b). The real-world graphs are shown in Table 1(c). Some of the real-world graphs are the same as those in the experiment of PowerGraph. And some additional real-world graphs are from the UF Sparse Matrices Collection [6].

Table 1: Datasets

(a) Synthetic graphs: $\alpha \geq \beta$ (b) Synthetic graphs: $\alpha < \beta$ (c) Real-world graphs

Alias	α	β	$ E $	Alias	α	β	$ E $	Alias	Graph	$ V $	$ E $
S1	2.2	2.2	71,334,974	S10	2.1	2.2	88,617,300	Tw	Twitter [11]	42M	1.47B
S2	2.2	2.1	88,305,754	S11	2.0	2.2	135,998,503	Arab	Arabic-2005 [6]	22M	0.6B
S3	2.2	2.0	134,881,233	S12	2.0	2.1	145,307,486	Wiki	Wiki [2]	5.7M	130M
S4	2.2	1.9	273,569,812	S13	1.9	2.2	280,090,594	LJ	LiveJournal [17]	5.4M	79M
S5	2.1	2.1	103,838,645	S14	1.9	2.1	289,002,621	WG	WebGoogle [13]	0.9M	5.1M
S6	2.1	2.0	164,602,848	S15	1.9	2.0	327,718,498				
S7	2.1	1.9	280,516,909								
S8	2.0	2.0	208,555,632								
S9	2.0	1.9	310,763,862								

5.2 Baselines and Evaluation Metric

In our experiment, we adopt the *Random* of PowerGraph [7] and the *Grid* of GraphBuilder [9]¹ as baselines for empirical comparison. The method *Hybrid* of PowerLyra [4] is not adopted for comparison because it combines both edge-cut and vertex-cut which is not a pure vertex-cut method.

One important metric is the replication factor, which reflects the communication cost. To test the speedup for real applications, we use the total execution time for *PageRank* which is forced to take 100 iterations. The speedup is defined as: $speedup = 100\% \times (\gamma_{Alg} - \gamma_{DBH}) / \gamma_{Alg}$, where γ_{Alg} is the execution time of PageRank with the method *Alg*. Here, *Alg* can be *Random* or *Grid*. Because all the methods can achieve good workload balance in our experiments, we do not report it here.

¹GraphLab 2.2 released in July 2013 has used PowerGraph as its engine, and the *Grid* GP method has been adopted by GraphLab 2.2 to replace the original *Random* GP method. Detailed information can be found at: <http://graphlab.org/projects/index.html>

5.3 Results

Figure 3 shows the replication factor on two subsets of synthetic graphs. We can find that our DBH method achieves much lower replication factor than *Random* and *Grid*. The replication factor of DBH is reduced by up to 80% compared to *Random* and 60% compared to *Grid*.

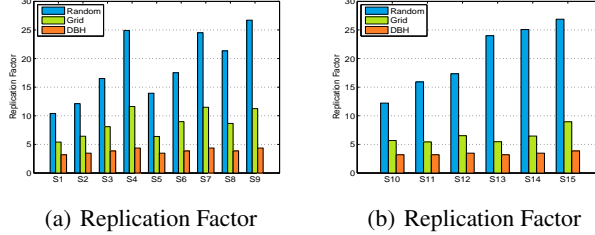


Figure 3: Experiments on two subsets of synthetic graphs. The X-axis denotes different datasets in Table 1(a) and Table 1(b). The number of machines is 48.

Figure 4 (a) shows the replication factor on the real-world graphs. We can also find that DBH achieves the best performance. Figure 4 (b) shows that the relative speedup of DBH is up to 60% over *Random* and 25% over *Grid* on the PageRank computation.

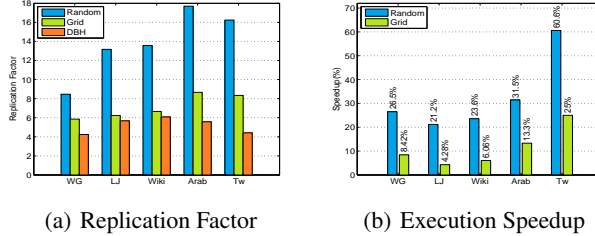


Figure 4: Experiments on real-world graphs. The number of machines is 48.

Figure 5 shows the replication factor and execution time for PageRank on Twitter graph when the number of machines ranges from 8 to 64. We can find our DBH achieves the best performance for all cases.

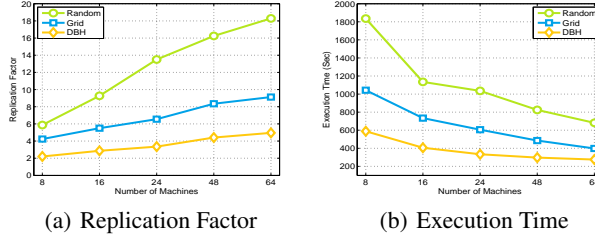


Figure 5: Experiments on Twitter graph. The number of machines ranges from 8 to 64.

6 Conclusion

In this paper, we have proposed a new vertex-cut graph partitioning method called *degree-based hashing* (DBH) for distributed graph-computing frameworks. DBH can effectively exploit the power-law degree distributions in natural graphs to achieve promising performance. Both theoretical and empirical results show that DBH can outperform the state-of-the-art methods. In our future work, we will apply DBH to more big data machine learning tasks.

7 Acknowledgements

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A Proofs

A.1 The Proof of Theorem 1

Proof. Let the indicator H_j denote the event that vertex v_i has at least one of h_i edges in the j th machine. Then the expectation $\mathbb{E}[H_j]$ is

$$\begin{aligned}\mathbb{E}[H_j] &= 1 - \Pr(\text{none of the } h_i \text{ edges is on the machine } j) \\ &= 1 - \left(1 - \frac{1}{p}\right)^{h_i}.\end{aligned}$$

For some vertex v_i , h_i adjacent edges are hashed by the neighbours of v_i and $(d_i - h_i)$ adjacent edges are hashed by v_i itself to the same machine. So for the $(d_i - h_i)$ edges, the number of replications of v_i is simply 1 due to the assumption $h_i \leq d_i - 1$.

As for the residual h_i of the adjacent edges, we have $\mathbb{E}[H_j] = 1 - \left(1 - \frac{1}{p}\right)^{h_i}$. Here H_j involves the other $p - 1$ machines except for the one that already has a replication.

Putting the two parts together, we have

$$\begin{aligned}\mathbb{E}[|A(v_i)|] &= 1 + \sum_{j=1}^{p-1} \mathbb{E}[H_j] = 1 + (p-1) \left[1 - \left(1 - \frac{1}{p}\right)^{h_i}\right] \\ &= p \left[1 - \left(1 - \frac{1}{p}\right)^{h_i+1}\right].\end{aligned}$$

Thus, the expected replication factor is:

$$\begin{aligned}\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n |A(v_i)|\right] &= \frac{1}{n} \sum_{i=1}^n \left[p \left(1 - \left(1 - \frac{1}{p}\right)^{h_i+1}\right)\right] \\ &= \frac{p}{n} \sum_{i=1}^n \left[1 - \left(1 - \frac{1}{p}\right)^{h_i+1}\right].\end{aligned}$$

□

Note that we assume $h_i \leq d_i - 1$, which means the hash function hashes at least one of the adjacent edges of v_i by v_i itself. Such assumption is to guarantee that there will be no “single” master vertex as which no adjacent edges are in the same partition.

By Lemma 1, we obtain

$$\frac{p}{n} \sum_{i=1}^n \left[1 - \left(1 - \frac{1}{p}\right)^{h_i+1}\right] \leq \frac{p}{n} \sum_{i=1}^n \left[1 - \left(1 - \frac{1}{p}\right)^{d_i}\right],$$

which implies that the hash-based vertex-cut via degree-approach is at least as good as the randomized vertex-cut in the replication factor.

A.2 The Proof of Theorem 2

Proof. Since we assume that the vertices are evenly hashed to all the machines, the h_i adjacent edges of v_i are also evenly assigned to all the machines. Subsequently, each machine has $\frac{h_i}{p}$ edges. Thus, we sum up all the vertices, obtaining $\sum_{i=1}^n \frac{h_i}{p}$.

For the rest $d_i - h_i$ adjacent edges of v_i , they are assigned to the same machine. So this part of edges incurs imbalance.

In the above procedure each edge is actually assigned twice. Thus, the final result is

$$\frac{\max_m |\{e \in E \mid M(e) = m\}|}{|E|/p} = \frac{\sum_{i=1}^n \frac{h_i}{p} + \max_{j \in [p]} \sum_{v_i \in P_j} (d_i - h_i)}{2|E|/p}.$$

□

Like Theorem 1, the result in Theorem 2 also depends on the choice of hash functions. For example, if a certain hash function has always hashed a constant number of all the adjacent edges of v_i by the v_i itself (which means $d_i - h_i = c$ is constant and $1 \leq c < \min_i d_i$), then the expected edge-imbalance is

$$\frac{\max_m |\{e \in E \mid M(e) = m\}|}{|E|/p} = \frac{\sum_{i=1}^n \frac{h_i}{p} + \max_{j \in [p]} \sum_{v_i \in P_j} (d_i - h_i)}{2|E|/p} = \frac{\sum_{i=1}^n \frac{d_i - c}{p} + c \frac{n}{p}}{2|E|/p} = 1$$

for $\sum_{i=1}^n d_i = 2|E|$.

Note that there must be a tradeoff between the replication factor and edge-imbalance. To make the replication factor smaller, h_i must be reduced. As a result, a smaller h_i yields larger edge-imbalance according to Theorems 1 and 2.

A.3 The Proof of Theorem 3

Proof. Before proving the following theorems, we introduce some properties of the power-law distribution. Note that although the degree distribution of a random graph should take integers, it is common to approximate the power-law degree distribution by continuous real numbers, which is suggested by [5]. Furthermore, although the maximum degree is $n - 1$, for convenience we assume that the maximum degree approaches infinity as n is very large.

Lemma 3. *The power-law distribution is defined as:*

$$\Pr(d = x) = p(x) = (\alpha - 1)x_{\min}^{\alpha-1}x^{-\alpha}, \quad \text{for } x \geq x_{\min}. \quad (2)$$

The corresponding CDF (cumulative distribution function) is

$$\Pr(d \leq x) = F(x) = 1 - \frac{x^{1-\alpha}}{x_{\min}^{1-\alpha}}. \quad (3)$$

The k th moment is

$$\mathbb{E}[x^k] = x_{\min} \times \frac{\alpha - 1}{\alpha - 1 - k}, \quad \text{for } \alpha > k + 1. \quad (4)$$

We then introduce an important inequality for our proof.

Lemma 4.

$$b^x \geq \theta x + c, \quad \text{for } b \in (0, 1), \quad x \geq 0, \quad \theta = b^\Omega \ln b, \quad c = b^\Omega - \Omega b^\Omega \ln b, \quad (5)$$

where Ω can be any positive value. Now we return to the proof.

Let $b = 1 - 1/p$. By Eqn. (5) we have

$$1 - \left(1 - \frac{1}{p}\right)^{d_i} \leq (1 - c) - \theta d_i.$$

Taking expectation on the degree sequence $D = \{d_i\}_{i=1}^n$ and by Eqn. (4), we get

$$\begin{aligned} \mathbb{E}_D \left[1 - \left(1 - \frac{1}{p}\right)^{d_i} \right] &\leq \mathbb{E}_D [(1 - c) - \theta d_i] \\ &= (1 - c) - \theta \mathbb{E}_D [d_i] = (1 - c) - \theta d_{\min} \times \frac{\alpha - 1}{\alpha - 2}. \end{aligned}$$

Since d_i are i.i.d, we obtain the result:

$$\mathbb{E}_D \left[\frac{p}{n} \sum_{i=1}^n \left(1 - \left(1 - \frac{1}{p}\right)^{d_i} \right) \right] = \frac{p}{n} \sum_{i=1}^n \mathbb{E}_D \left[1 - \left(1 - \frac{1}{p}\right)^{d_i} \right] \leq p \left[1 - c - \theta d_{\min} \times \frac{\alpha - 1}{\alpha - 2} \right],$$

where $c = (1 - \frac{1}{p})^\Omega - \Omega(1 - \frac{1}{p})^{\Omega-1} \ln(1 - \frac{1}{p})$ and $\theta = (1 - \frac{1}{p})^\Omega \ln(1 - \frac{1}{p})$. We take the derivative with respect to Ω to get the tightest bound. Thus when $\Omega = d_{\min}(\frac{\alpha-1}{\alpha-2})$, the derivative is 0 and the second order derivative on this point is positive. Note that the derivative is positive if $\Omega > d_{\min}(\frac{\alpha-1}{\alpha-2})$ and negative if $\Omega < d_{\min}(\frac{\alpha-1}{\alpha-2})$. So $\hat{\Omega} = d_{\min}(\frac{\alpha-1}{\alpha-2})$ is a global minimizer. Finally we can have:

$$\mathbb{E}_D \left[\frac{p}{n} \sum_{i=1}^n \left(1 - \left(1 - \frac{1}{p}\right)^{d_i} \right) \right] \leq p \left[1 - \left(1 - \frac{1}{p}\right)^{\hat{\Omega}} \right],$$

where $\hat{\Omega} = d_{\min} \times \frac{\alpha-1}{\alpha-2}$. □

A.4 The Proof of Theorem 4

Proof. First we view the two sequences $D = \{d_i\}_{i=1}^n$ and $H = \{h_i\}_{i=1}^n$ as constant values. By Eqn. (5) we have:

$$1 - \left(1 - \frac{1}{p}\right)^{h_i+1} \leq (1-c) - \theta(h_i+1).$$

Then we take the expectation on $\{h_i\}_{i=1}^n$ conditional on $\{d_i\}_{i=1}^n$, leading to

$$\begin{aligned} \mathbb{E}_H \left[1 - \left(1 - \frac{1}{p}\right)^{h_i+1} \middle| D \right] &\leq \mathbb{E}_H [(1-c) - \theta(h_i+1) | D] \\ &= (1-c) - \theta \mathbb{E}_H [(h_i+1) | D]. \end{aligned}$$

Note that under our assumption, each of the adjacent edges is chosen to be hashed by v_i with probability $\Pr(d \geq d_i) = 1 - \Pr(d \leq d_i) = \frac{d_i^{1-\alpha}}{d_{min}^{1-\alpha}}$. Thus, by the assumption $h_i \leq d_i - 1$, we have

$$\mathbb{E}_H [d_i - h_i | D] = 1 + (d_i - 1) \times \frac{d_i^{1-\alpha}}{d_{min}^{1-\alpha}},$$

which means

$$\mathbb{E}_H [h_i + 1 | D] = d_i - (d_i - 1) \times \frac{d_i^{1-\alpha}}{d_{min}^{1-\alpha}} = d_i - \frac{d_i^{2-\alpha}}{d_{min}^{1-\alpha}} + \frac{d_i^{1-\alpha}}{d_{min}^{1-\alpha}}.$$

Then we further take the expectation on $D = \{d_i\}_{i=1}^n$:

$$\begin{aligned} \mathbb{E}_D \left[d_i - \frac{d_i^{2-\alpha}}{d_{min}^{1-\alpha}} + \frac{d_i^{1-\alpha}}{d_{min}^{1-\alpha}} \right] &= \int_{d_{min}}^{+\infty} \left(d_i - \frac{d_i^{2-\alpha}}{d_{min}^{1-\alpha}} + \frac{d_i^{1-\alpha}}{d_{min}^{1-\alpha}} \right) (\alpha-1) d_{min}^{\alpha-1} d^{-\alpha} \mathbf{d} d_i \\ &= d_{min} \times \frac{\alpha-1}{\alpha-2} - d_{min} \times \frac{\alpha-1}{2\alpha-3} + \frac{1}{2}. \end{aligned}$$

Finally, we obtain the result:

$$\begin{aligned} \mathbb{E}_{H,D} \left[\frac{p}{n} \sum_{i=1}^n \left(1 - \left(1 - \frac{1}{p}\right)^{h_i+1} \right) \right] &= \frac{p}{n} \sum_{i=1}^n \mathbb{E}_{H,D} \left[1 - \left(1 - \frac{1}{p}\right)^{h_i+1} \right] \\ &\leq p \left[1 - c - \theta \left(d_{min} \times \frac{\alpha-1}{\alpha-2} - d_{min} \times \frac{\alpha-1}{2\alpha-3} + \frac{1}{2} \right) \right]. \end{aligned}$$

Here $c = \left(1 - \frac{1}{p}\right)^\Omega - \Omega \left(1 - \frac{1}{p}\right)^\Omega \ln \left(1 - \frac{1}{p}\right)$ and $\theta = \left(1 - \frac{1}{p}\right)^\Omega \ln \left(1 - \frac{1}{p}\right)$. The expectation above is taken w.r.t. both $\{h_i\}_{i=1}^n$ and $\{d_i\}_{i=1}^n$.

Similar to Theorem 3, we take derivative w.r.t. Ω and get the global minimizer $\hat{\Omega}' = \left(d_{min} \times \frac{\alpha-1}{\alpha-2} - d_{min} \times \frac{\alpha-1}{2\alpha-3} + \frac{1}{2} \right)$. Thus we get the tightest bound:

$$\mathbb{E}_{H,D} \left[\frac{p}{n} \sum_{i=1}^n \left(1 - \left(1 - \frac{1}{p}\right)^{h_i+1} \right) \right] \leq p \left[1 - \left(1 - \frac{1}{p}\right)^{\hat{\Omega}'} \right],$$

where $\hat{\Omega}' = \left(d_{min} \times \frac{\alpha-1}{\alpha-2} - d_{min} \times \frac{\alpha-1}{2\alpha-3} + \frac{1}{2} \right)$.

Note that our assumption suggests $d_{min} \geq 2$. And for $\alpha \in (2, 3)$, $-d_{min} \frac{\alpha-1}{2\alpha-3} + \frac{1}{2} < 0$. Thus we have

$$\left(d_{min} \times \frac{\alpha-1}{\alpha-2} - d_{min} \times \frac{\alpha-1}{2\alpha-3} + \frac{1}{2} \right) < d_{min} \times \frac{\alpha-1}{\alpha-2}.$$

Thus we have:

$$p \left[1 - \left(1 - \frac{1}{p}\right)^{\hat{\Omega}'} \right] < p \left[1 - \left(1 - \frac{1}{p}\right)^{\hat{\Omega}} \right].$$

□

A.5 The Proof of Theorem 5

Proof. Define $x_i = (d_i - h_i)$. If $\{d_i\}_{i=1}^n$ is fixed, then $x_i \in [1, d_i]$. For specific $j \in [p]$, by Hoeffding's tail inequality, we get the following inequality conditional on the sequence $D = \{d_i\}_{i=1}^n$:

$$\Pr_H \left\{ \sum_{v_i \in P_j} x_i \geq \mathbb{E}_H \left[\sum_{v_i \in P_j} x_i \right] + t \mid D \right\} \leq \exp \left(\frac{-2t^2}{\sum_{v_i \in P_j} (d_i - 1)^2} \right), \text{ for any } t > 0.$$

Thus we obtain the inequality of the maximum value of the p individual sums:

$$\Pr_H \left\{ \max_{j \in [p]} \sum_{v_i \in P_j} x_i \leq \frac{n}{p} \mathbb{E}_H[x_i] + t \mid D \right\} \geq \left[1 - \exp \left(\frac{-2t^2}{\sum_{v_i \in P_j} (d_i - 1)^2} \right) \right]^p.$$

Here we take $t = \epsilon \mathbb{E}_D \left[\sum_{v_i \in P_j} d_i \right] = \epsilon \frac{n}{p} \mathbb{E}[d]$ where $\epsilon \in (0, 1)$. Then we have

$$\sum_{v_i \in P_j} \frac{(d_i - 1)^2}{t^2} = \sum_{v_i \in P_j} \frac{(d_i - 1)^2}{\mathbb{E}^2[d] \epsilon^2 n^2 / p^2}.$$

Note that for $v_i \in P_j$, $\{d_i - 1\}_{i=1}^n$ is also a sequence of values under power-law distribution. When $n \rightarrow \infty$, the number of any specific value of degree $\#d_i \asymp n \Pr(d_i)$. Thus we have

$$\lim_{n \rightarrow \infty} \sum_{v_i \in P_j} \frac{(d_i - 1)^2}{n^2 / p^2} \asymp \lim_{n \rightarrow \infty} \int_{d_{\min}}^{n-1} \frac{(d-1)^2 n \Pr(d)}{n^2} \mathbf{d}d.$$

By our assumption $\alpha \in (2, 3)$ and $\Pr(d) \propto d^{-\alpha}$,

$$\lim_{n \rightarrow \infty} \int_{d_{\min}}^{n-1} \frac{(d-1)^2 n \Pr(d)}{n^2} \mathbf{d}d = 0.$$

Thus for n large enough, under our choice of t , $\left(1 - \exp \left(-2t^2 / \sum_{v_i \in P_j} (d_i - 1)^2 \right) \right)^p$ is nearly 1. That is,

$\max_{j \in [p]} \sum_{v_i \in P_j} x_i \leq \frac{n}{p} \mathbb{E}[x_i] + t$ w.h.p. (with high probability) when n is very large. Namely,

$$\max_{j \in [p]} \sum_{v_i \in P_j} x_i \leq \frac{n}{p} \mathbb{E}_H[x_i] + t = \frac{n}{p} \mathbb{E}_H[x_i] + \epsilon \frac{n}{p} \mathbb{E}[d].$$

Now we return to the imbalance factor. By using the result above, the following is gained w.h.p:

$$\sum_{i=1}^n \frac{h_i}{p} + \max_{j \in [p]} \sum_{v_i \in P_j} (d_i - h_i) \leq \sum_{i=1}^n \frac{h_i}{p} + \frac{n}{p} \mathbb{E}_H[x_i] + \epsilon \frac{n}{p} \mathbb{E}[d].$$

Then we take the expectation w.r.t. $\{h_i\}_{i=1}^n$ and $\{d_i\}_{i=1}^n$:

$$\begin{aligned} & \mathbb{E}_{H,D} \left[\sum_{i=1}^n \frac{h_i}{p} + \max_{j \in [p]} \sum_{v_i \in P_j} (d_i - h_i) \right] \\ & \leq \mathbb{E}_{H,D} \left[\sum_{i=1}^n \frac{h_i}{p} \right] + \frac{n}{p} \mathbb{E}_{H,D}[d_i - h_i] + \epsilon \frac{n}{p} \mathbb{E}_{H,D}[d] \\ & = (1 + \epsilon)n \mathbb{E}_{H,D}[d] / p \\ & = (1 + \epsilon)n \mathbb{E}_D[d] / p \end{aligned}$$

which completes the proof. \square