Balanced Binary Search Tree

Algorithm : Design & Analysis

[8]

In the last class…

- Finding *max* and *min*
- Finding the second largest key
- Adversary argument and lower bound
- Selection Problem – Median
- A Linear Time Selection Algorithm
- Analysis of Selection Algorithm
- A Lower Bound for Finding the Median

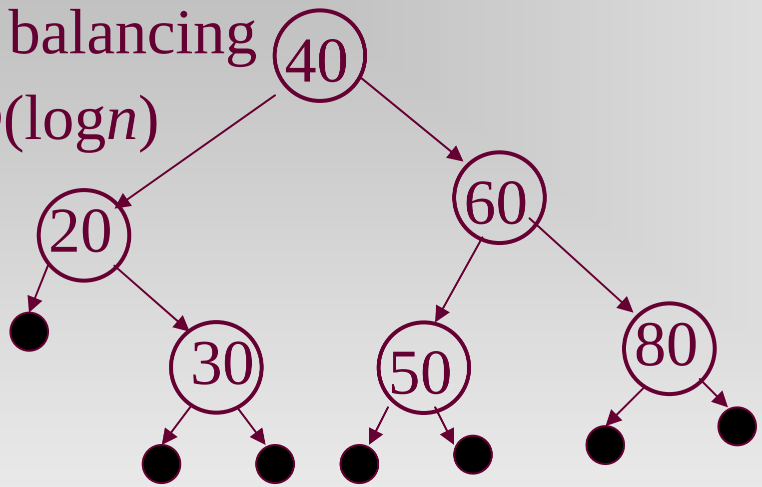
Balanced Binary Search Tree

- Definition of red-black tree
- Black height
- Insertion into a red-black tree
- Deletion from a red-black tree

Binary Search Tree Revisited

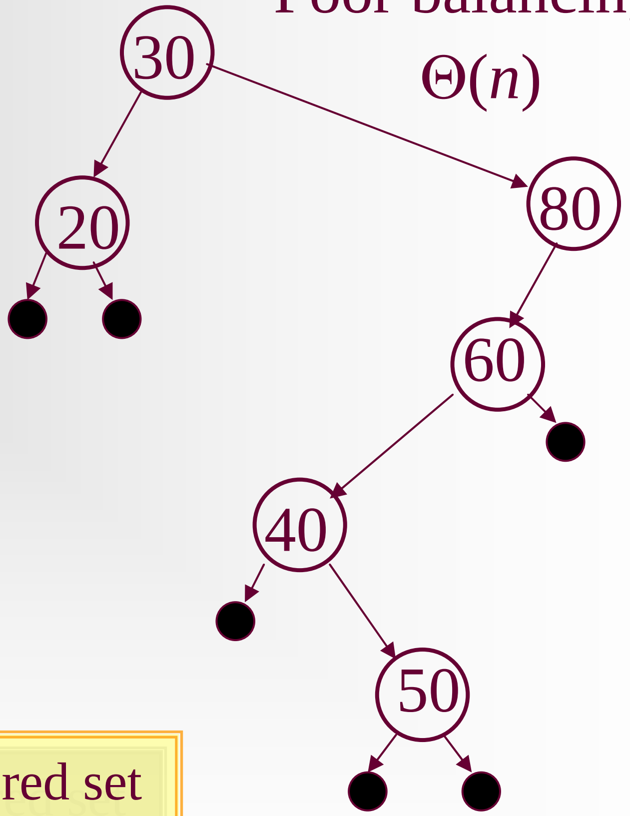
Good balancing

$\Theta(\log n)$



Poor balancing

$\Theta(n)$

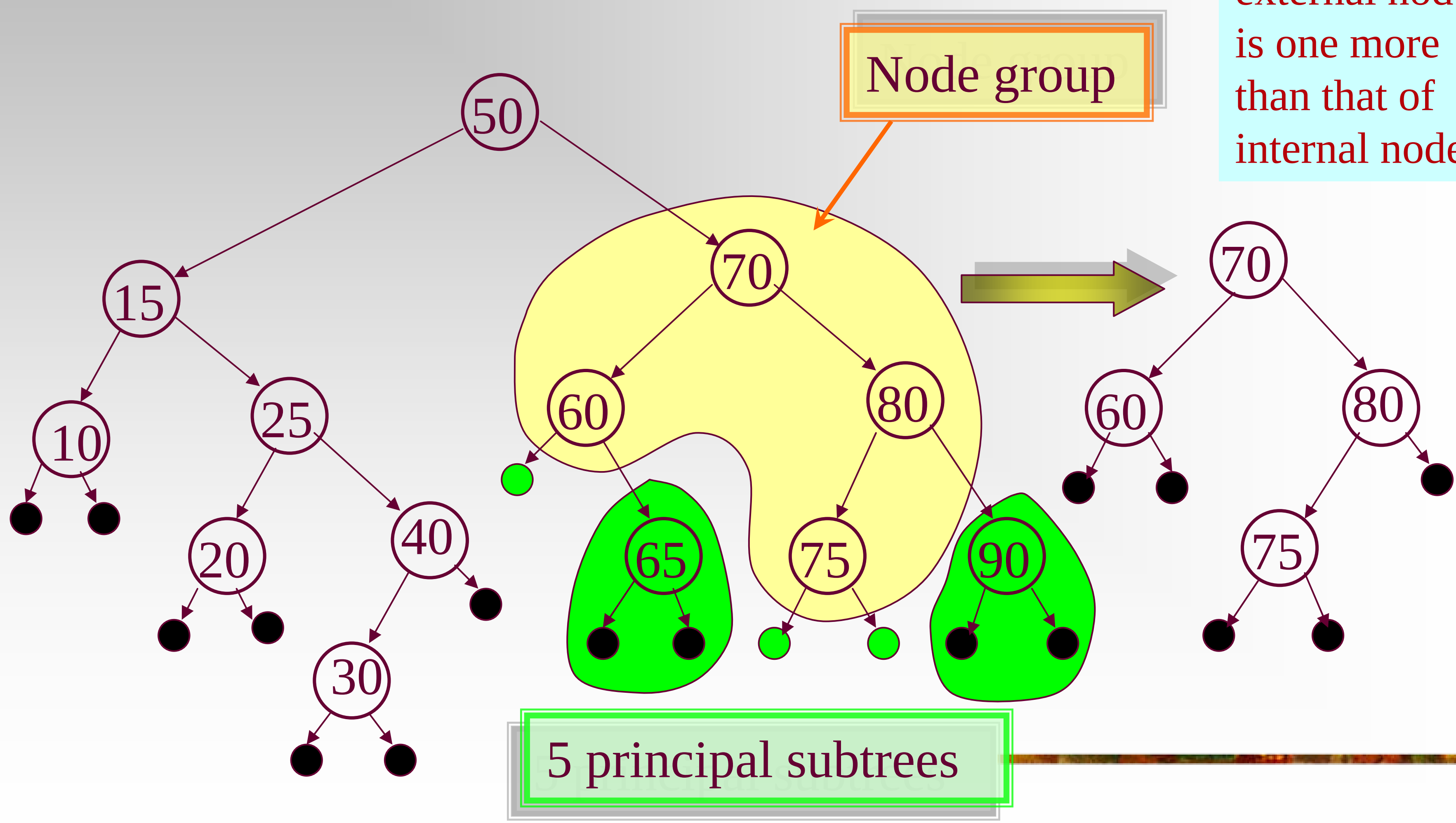


In a properly drawn tree, pushing forward to get the ordered list.

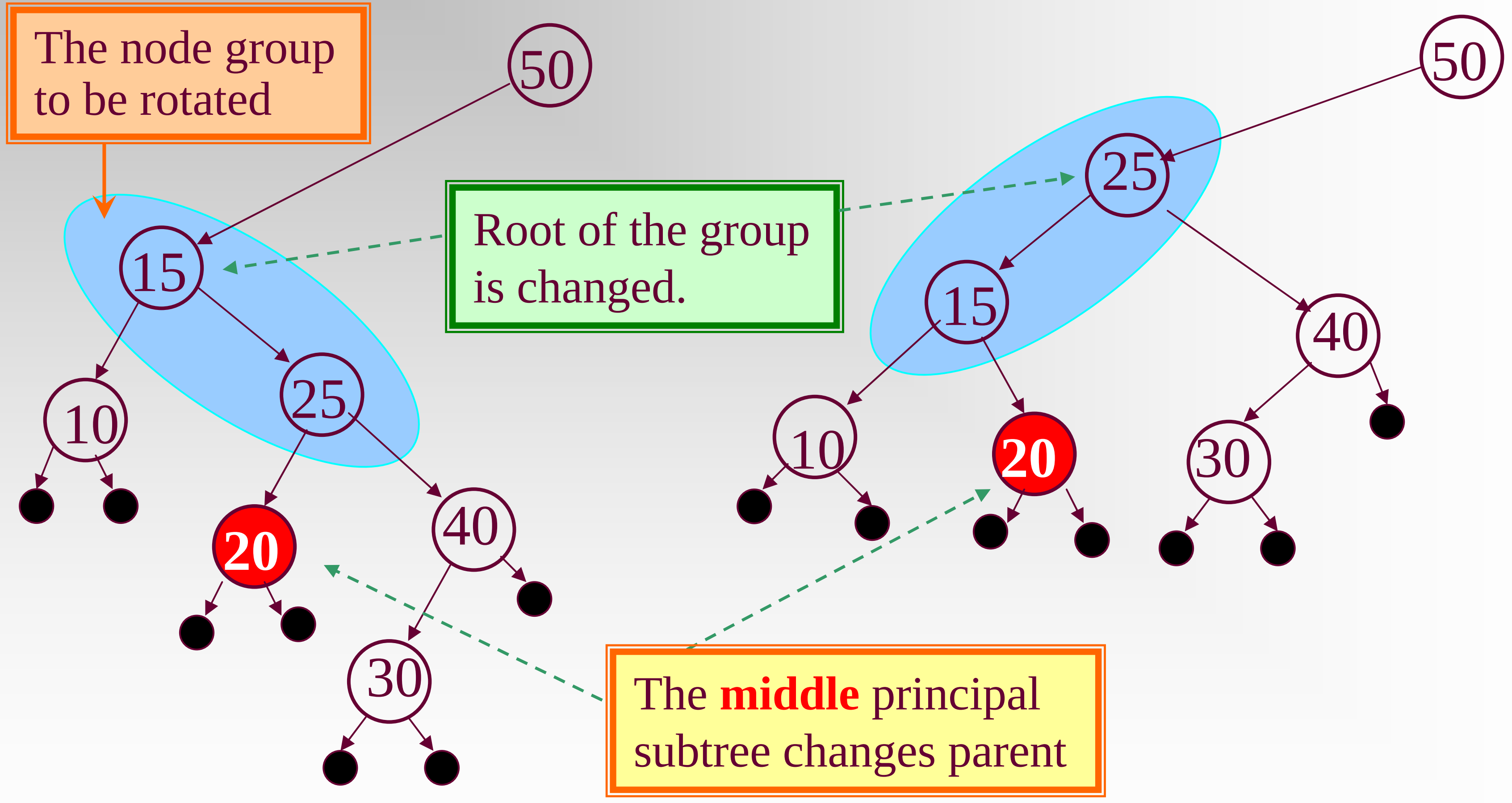
- Each node has a key, belonging to a linear ordered set
- An inorder traversal produces a sorted list of the keys

Node Group in a binTree

As in 2-tree, the number of external node is one more than that of internal node



Improving the Balancing by Rotation



Red-Black Tree: the Definition

- If *T* is a **binary tree** in which each node has a color, red or black, and all external nodes are black, then *T* is a **red-black tree** if and only if:
  - [*Color constraint*] No red node has a red child
  - [*Black height constrain*] The **black length** of all external paths from a given node *u* is the same (the black height of *u*)
  - The root is black.

Balancing is under controlled

- *Almost*-red-black tree(ARB tree)
  - Root is red, satisfying the other constraints.

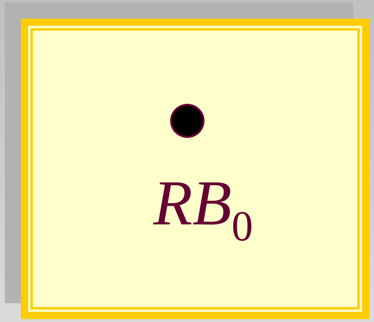
Recursive Definition of Red-Black Tree

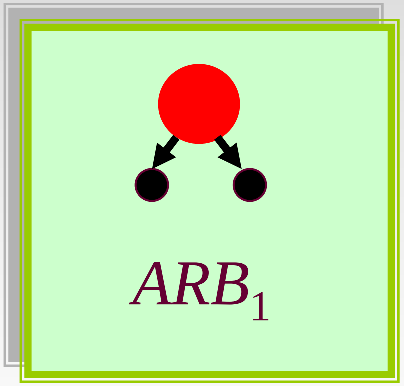
(A red-black tree of black height h is denoted as RB_h)

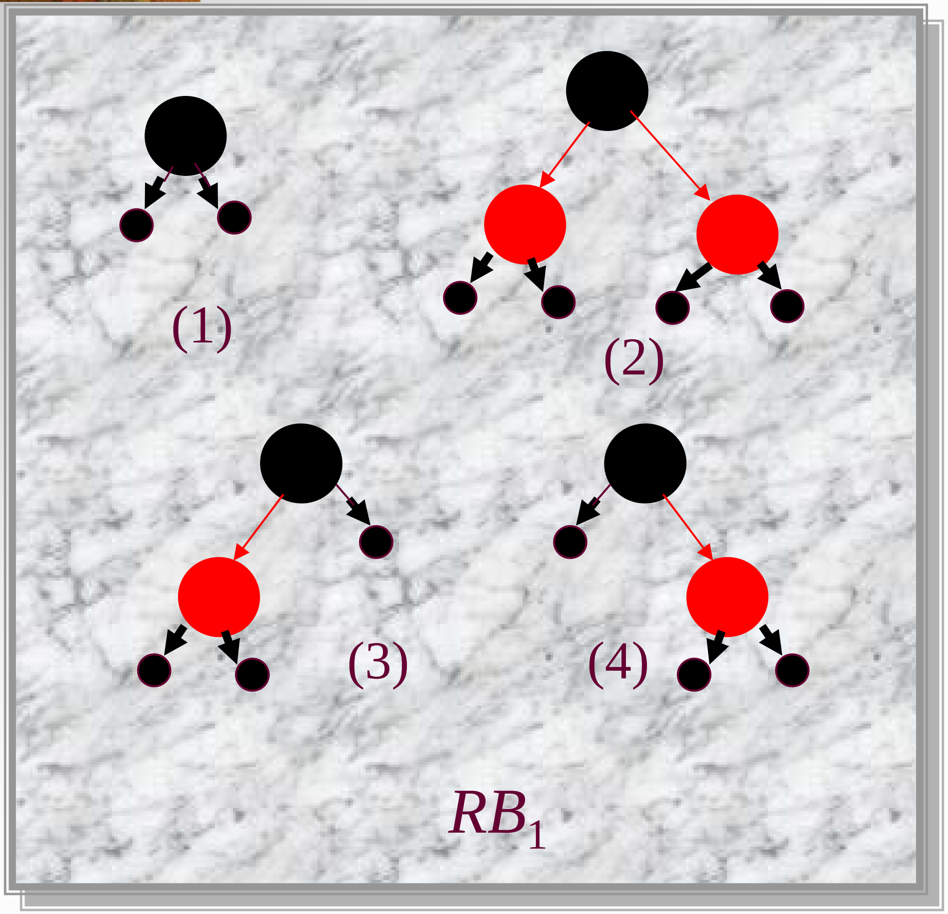
- Definition:
  - An external node is an RB_0 tree, and the node is black.
  - A binary tree is an ARB_h ($h \geq 1$) tree if:
    - Its root is red, and
    - Its left and right subtrees are each an RB_{h-1} tree.
  - A binary tree is an RB_h ($h \geq 1$) tree if:
    - Its root is black, and
    - Its left and right subtrees are each either an RB_{h-1} tree or an ARB_h tree.

No ARB_0

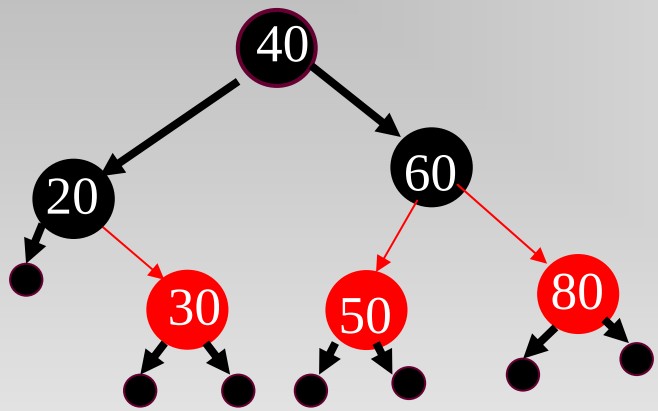
RB_i and ARB_i

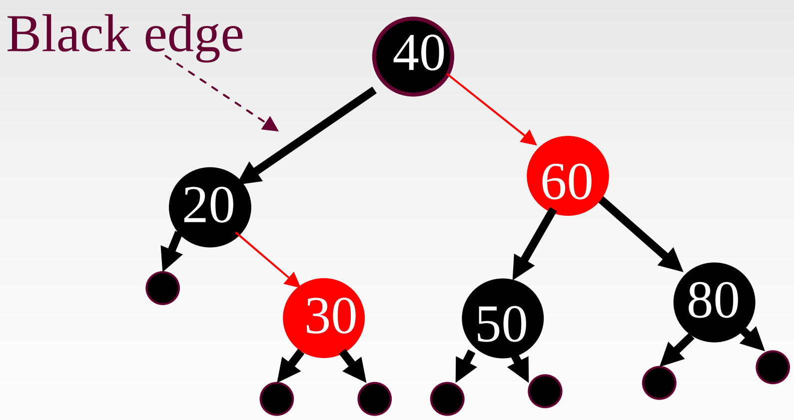




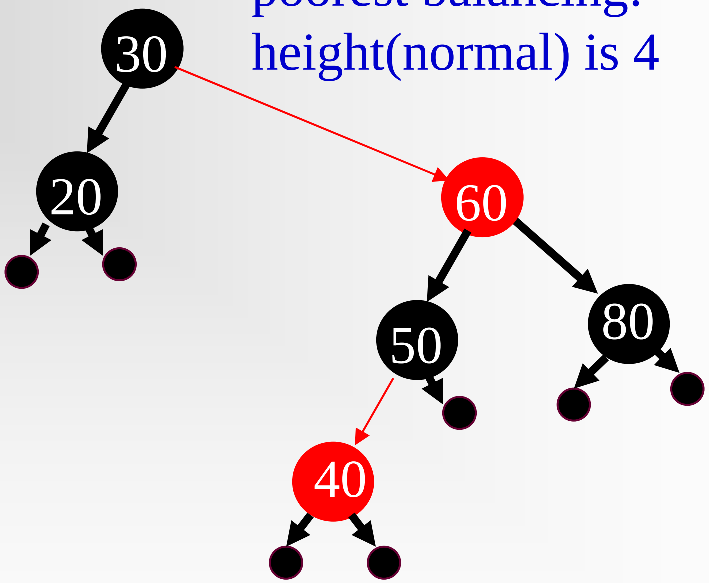


Red-Black Tree with 6 Nodes

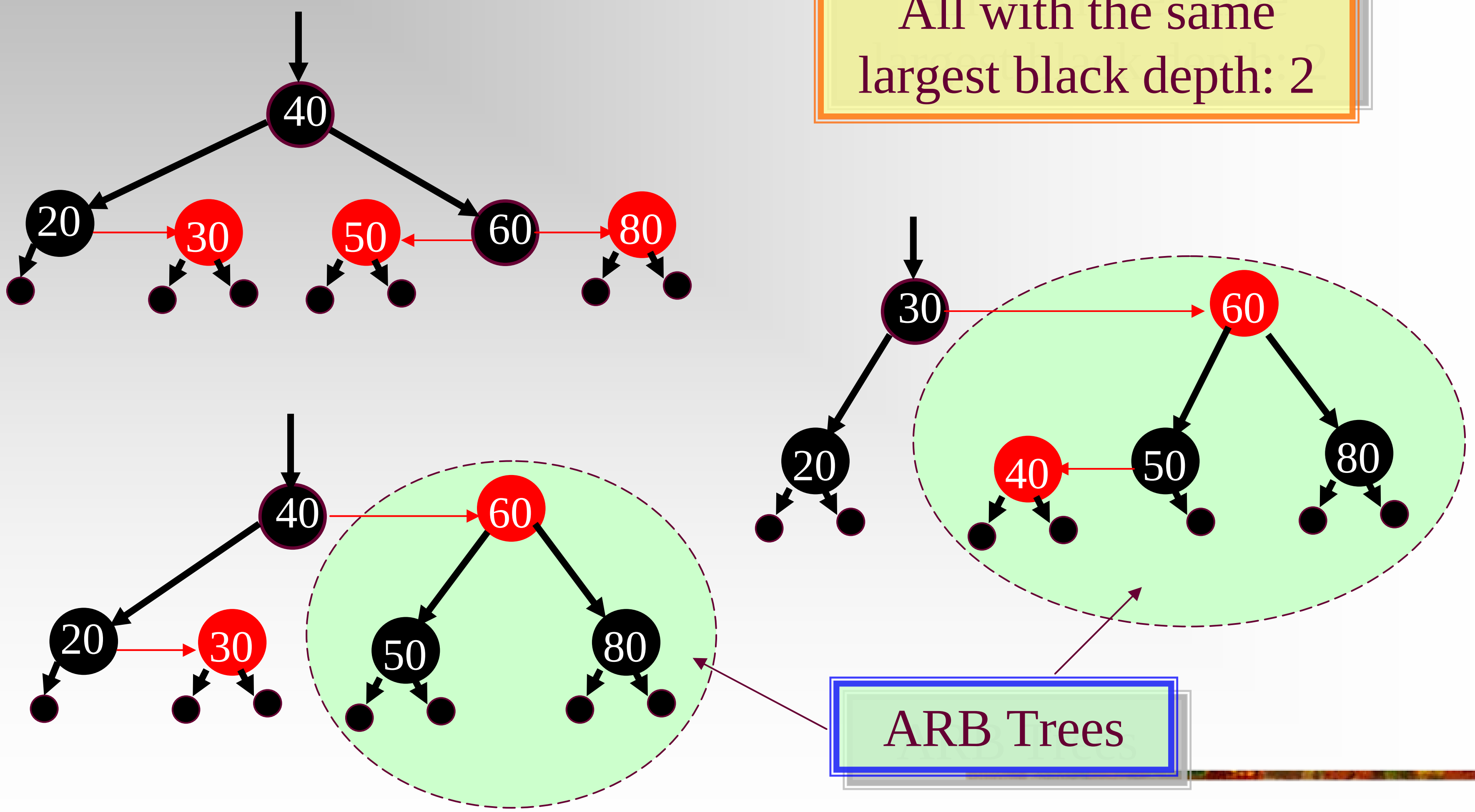




poorest balancing:
height(normal) is 4



Black-Depth Convention



Properties of Red-Black Tree

- The **black height** of any RB_h tree or ARB_h tree is well defind and is h.
- Let T be an RB_h tree, then:
  - T has at least 2^h-1 internal black nodes.
  - T has at most 4^h-1 internal nodes.
  - The depth of any black node is at most twice its black depth.
- Let A be an ARB_h tree, then:
  - A has at least 2^h-2 internal black nodes.
  - A has at most $(4^h)/2$-1 internal nodes.
  - The depth of any black node is at most twice its black depth.

Well-Defined Black Height

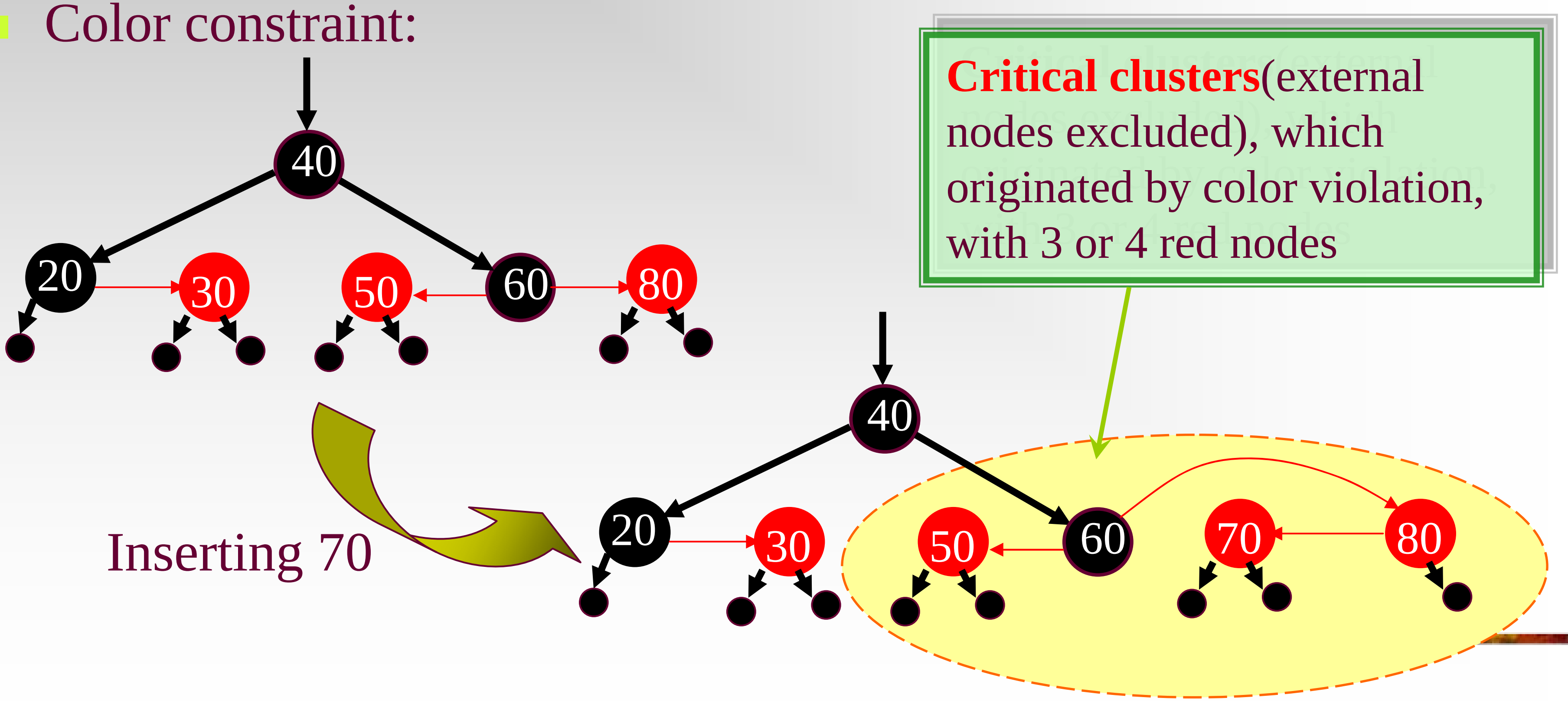
- That "the **black height** of any RB_h tree or ARB_h tree is well defind" means ***the black length of all external paths from the root is the same***.
- Proof: induction on h
- Base case: h=0, that is RB_0 (there is no ARB_0)
- In ARB_{h+1}, its two subtrees are both RB_h.Since the root is red, the black length of all external paths from the root is h, that's the same as its two subtrees.
- In RB_{h+1}:
  - Case 1: two subtrees are RB_h's
  - Case 2: two subtrees are ARB_{h+1}'s
  - Case 3: one subtree is an RB_h(black height=h), and the another is an ARB_{h+1}(black height=h+1)

Bound on Depth of Node in RBTree

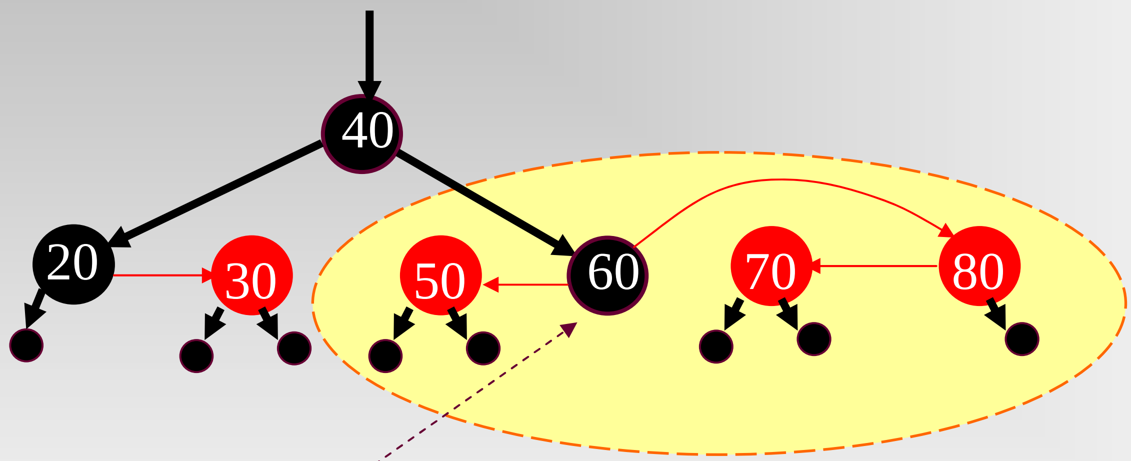
- Let T be a red-black tree with n internal nodes. Then no node has depth greater than $2\lg(n+1)$, which means that the height of T in the usual sense is at most $2\lg(n+1)$.
  - Proof:
  - Let h be the black height of T. The number of internal nodes, n, is at least the number of internal black nodes, which is at least 2^h-1, so $h \leq \lg(n+1)$. The node with greatest depth is some external node. All external nodes are with black depth h. So, the depth is at most $2h$.

Influences of Insertion into an RB Tree

- Black height constrain:
  - No violation ***if*** inserting a red node.
- Color constraint:

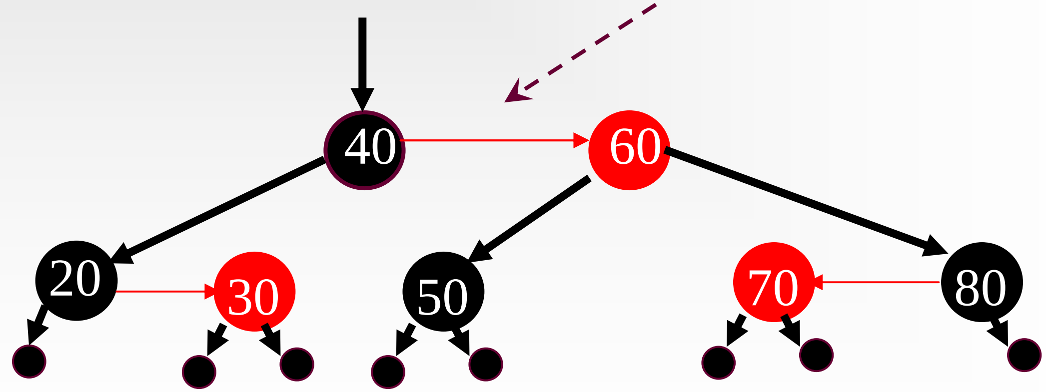


Repairing 4-node Critical Cluster

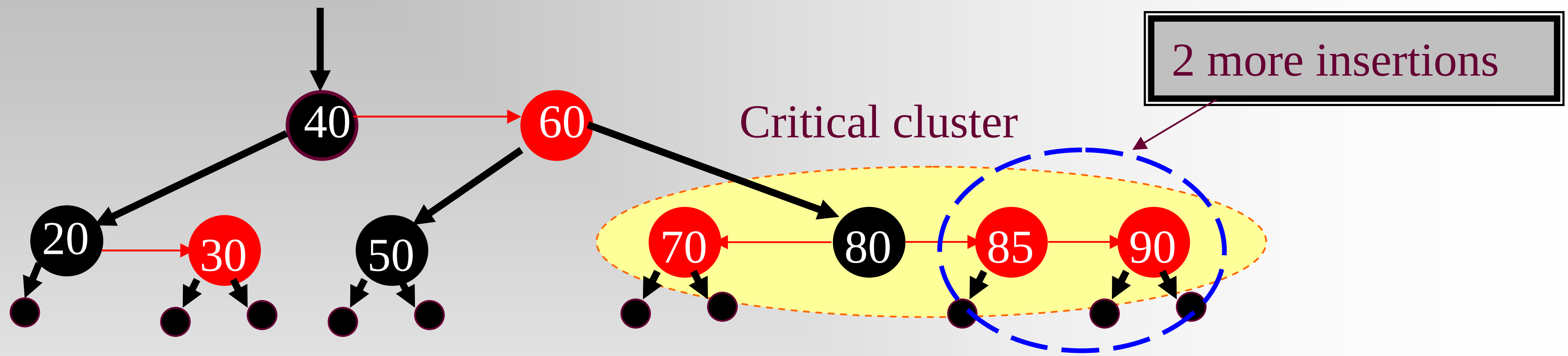


No new critical cluster occurs, inserting finished.

Color flip:
Root of the critical cluster exchanges color with its subtrees



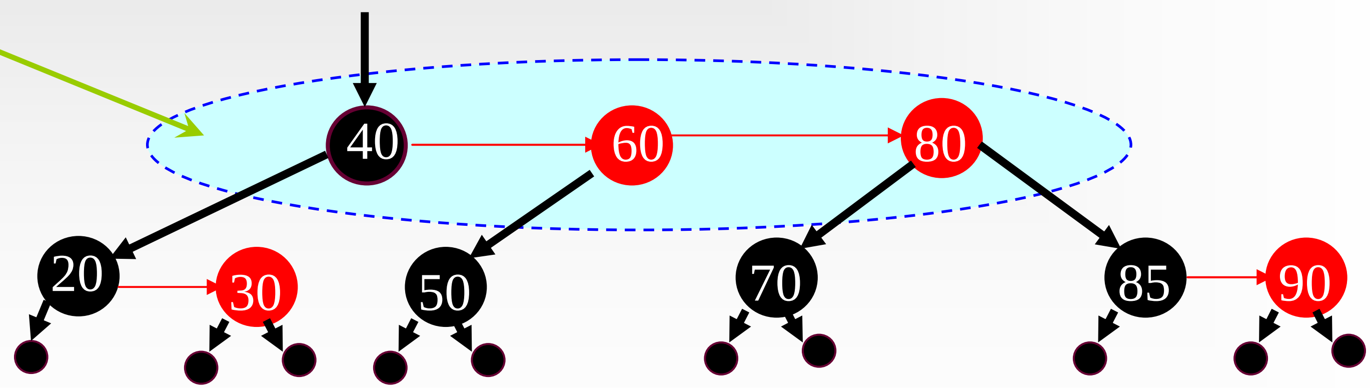
Repairing 4-node Critical Cluster



New critical cluster with 3 nodes.

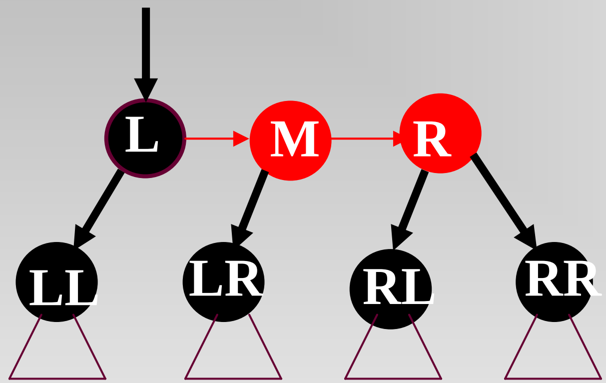
Color flip doesn't work,

Why?

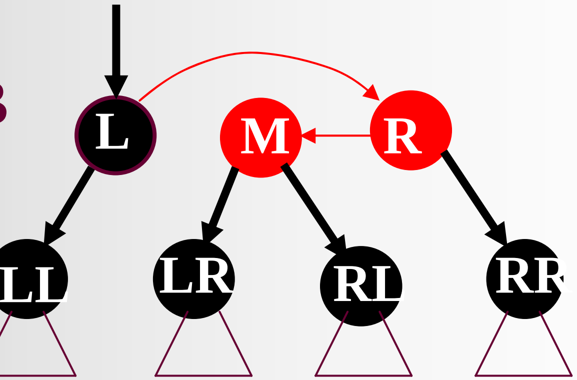


Patterns of 3-Node Critical Cluster

A

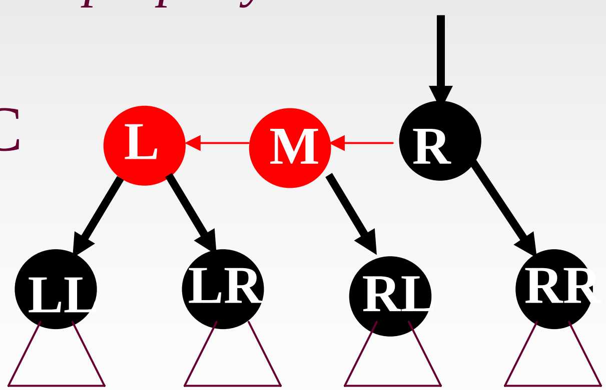


B



Shown as properly drawn

C



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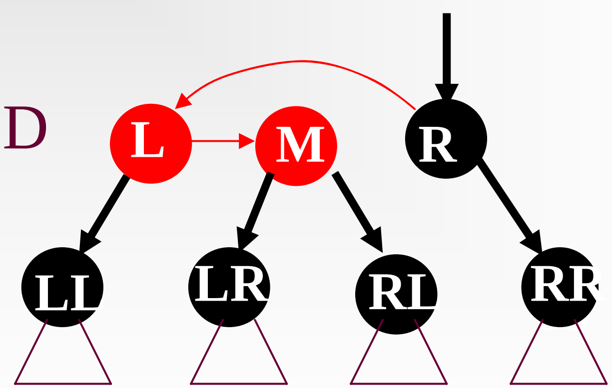
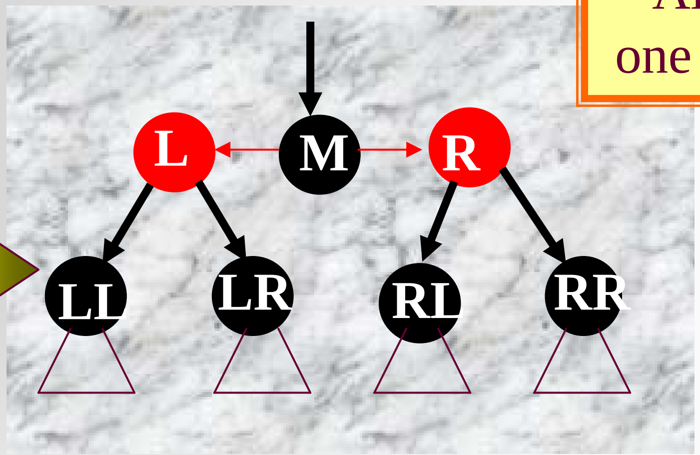


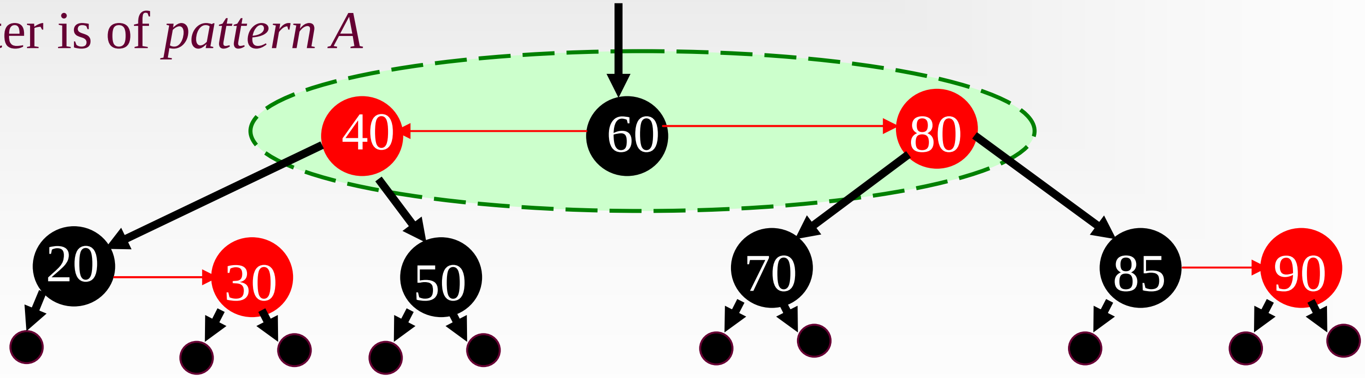
Repairing 3-Node Critical Cluster

All into
one pattern

Root of the critical cluster is changed to ***M***, and the parentship is adjusted accordingly



The incurred critical cluster is of *pattern A*



Implementing Insertion: Class

```
class RBtree
        Element root;
        RBtree leftSubtree;
        RBtree rightSubtree;
        int color; /* red, black */

        static class InsReturn
                public RBtree newTree;
                public int status /* ok, rbr, brb, rrb, brr */
```

Color pattern

Implementing Insertion: Procedure

```
RBtree rbtInsert (RBtree oldRBtree, Element newNode)
    InsReturn ans = rbtIns(oldREtree, newNode);
    If (ans.newT
        ans.newTr
    return ans.n
```

the wrapper

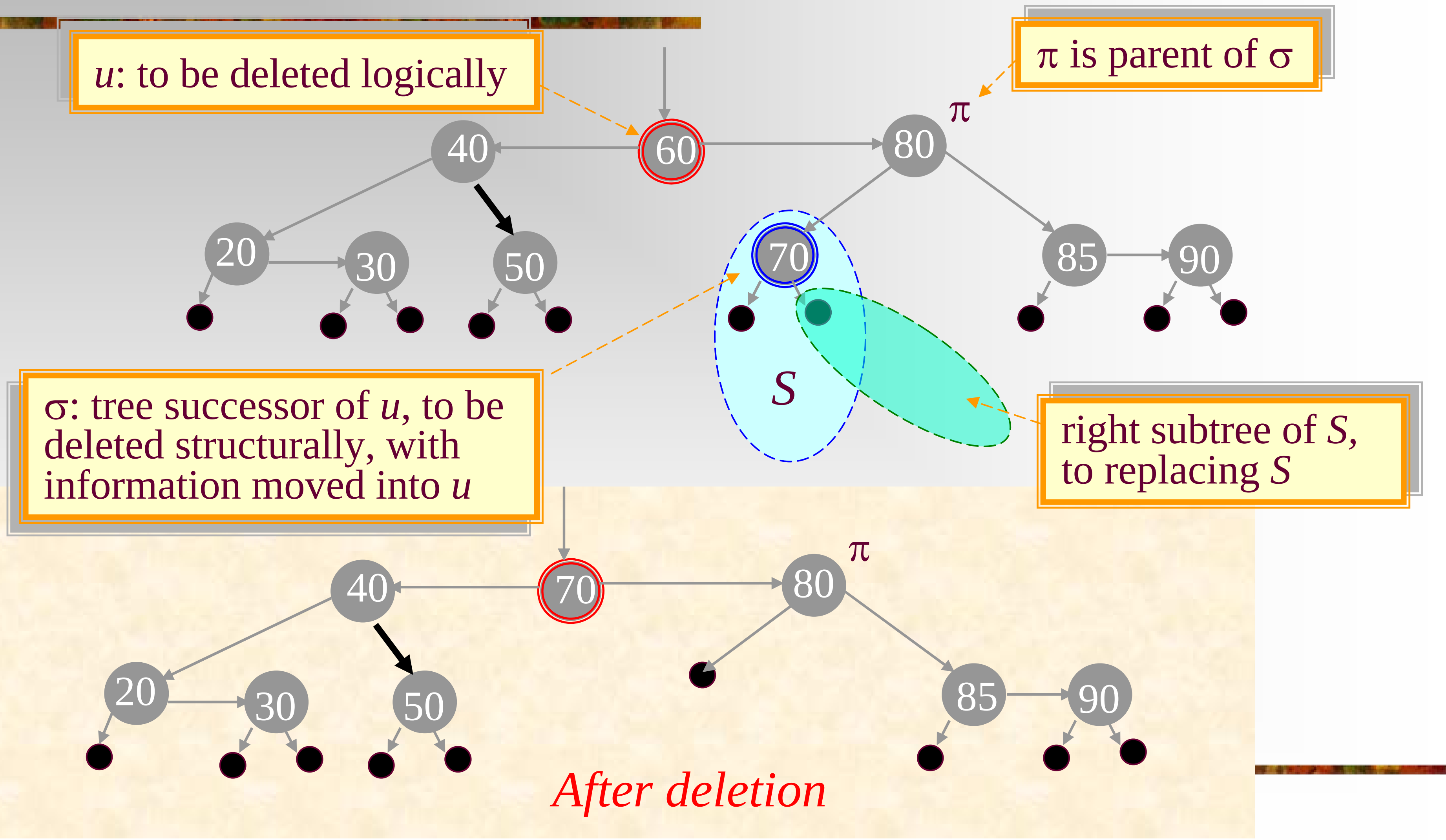
```
InsReturn rbtIns(RBtree oldRBtree, Element newNode)
    InsReturn ans, ansLeft, ansRight;
    if (oldRBtree = nil) then <Inserting simply>;
    else
        if (newNode.key <oldRBtree.root.key)
            ansLeft = rbtIns (oldRBtree.leftSubtree, newNode);
            ans = repairLeft(oldRBtree, ansLeft);
        else
            ansRight = rbtIns(oldRBtree.rightSubtree, newNode);
            ans = repairRight(oldRBtree, ansRight);
    return ans
```

the recursive function

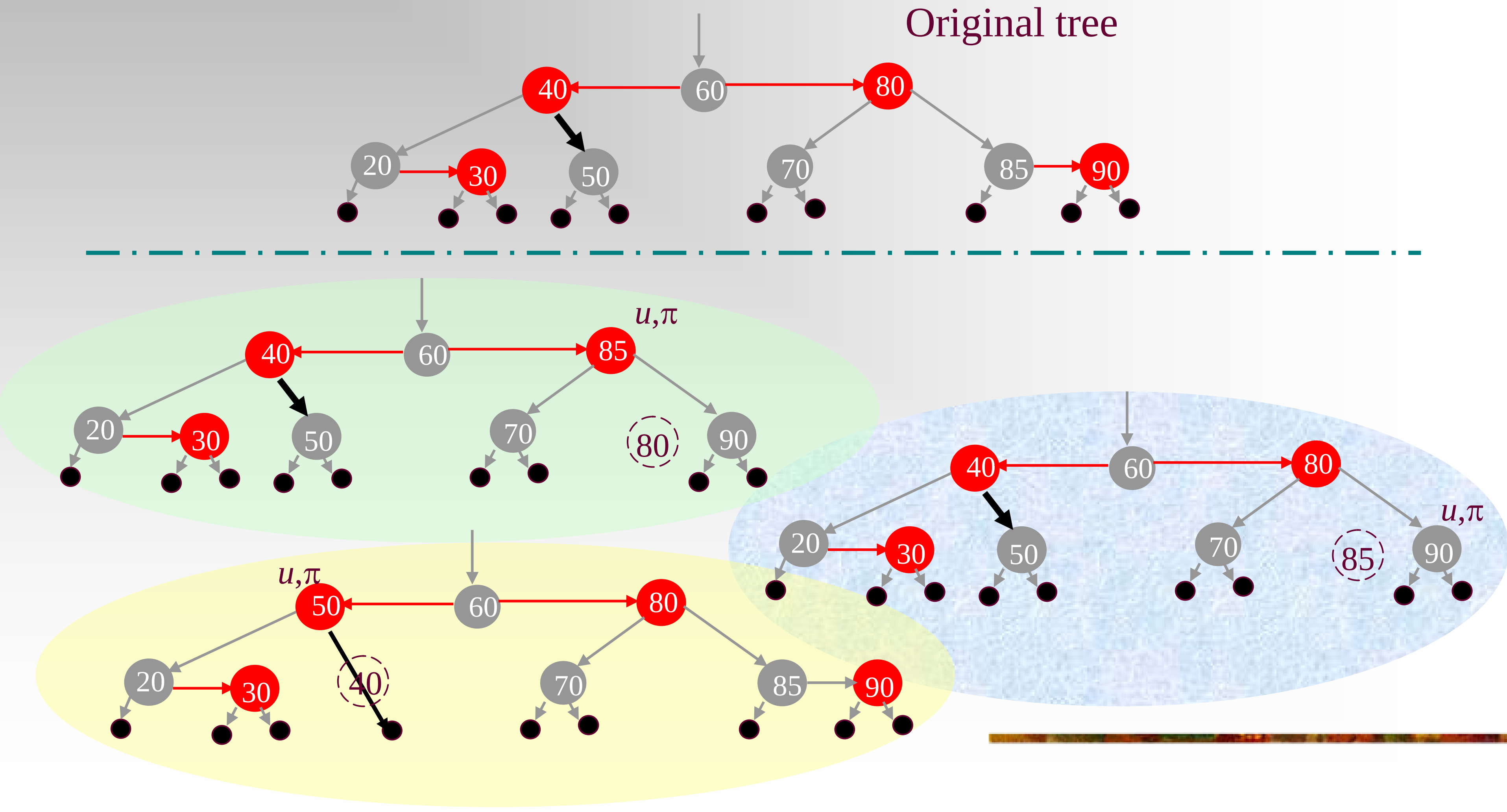
Correctness of Insertion

- If the parameter oldRBtree of rbtIns is an RB_h tree or an ARB_{h+1} tree(which is true for the recursive calls on rbtIns), then the newTree and status fields returned are one of the following combinations:
  - Status=ok, and newTree is an RB_h or an ARB_{h+1} tree,
  - Status=rbr, and newTree is an RB_h,
  - Status=brb, and newTree is an ARB_{h+1} tree,
  - Status=rrb, and newTree.color=red, newTree.leftSubtree is an ARB_{h+1} tree and newTree.rightSubtree is an RB_h tree,
  - Status=brr, and newTree.color=red, newTree.rightSubtree is an ARB_{h+1} tree and newTree.leftSubtree is an RB_h tree
- For those cases with red root, the color will be changed to black, with other constraints satisfied by repairing subroutines.

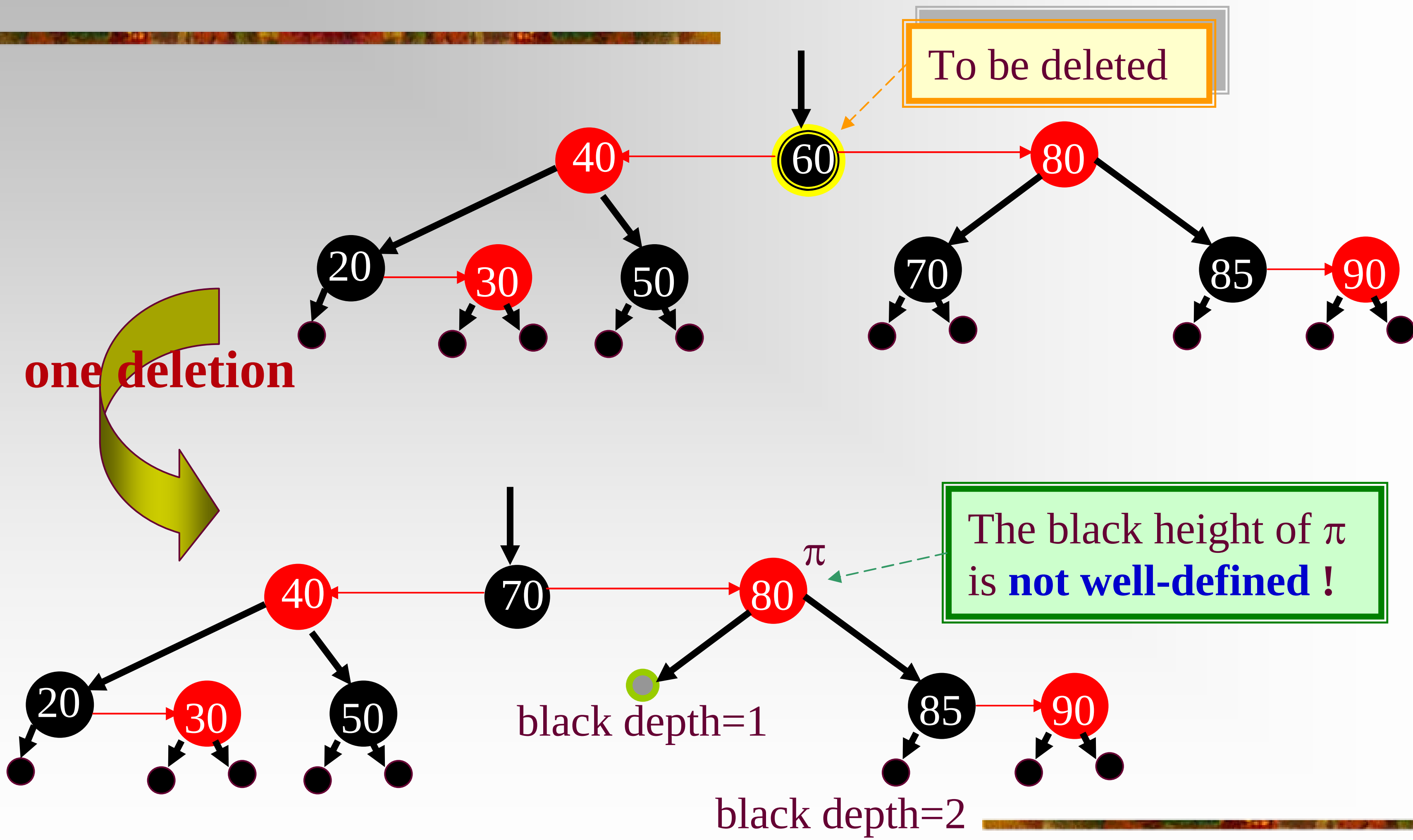
Deletion: Logical and Structral



Deletion from RBTree: Examples



Deletion in a Red-Black Tree



Procedure of Red-Black Deletion

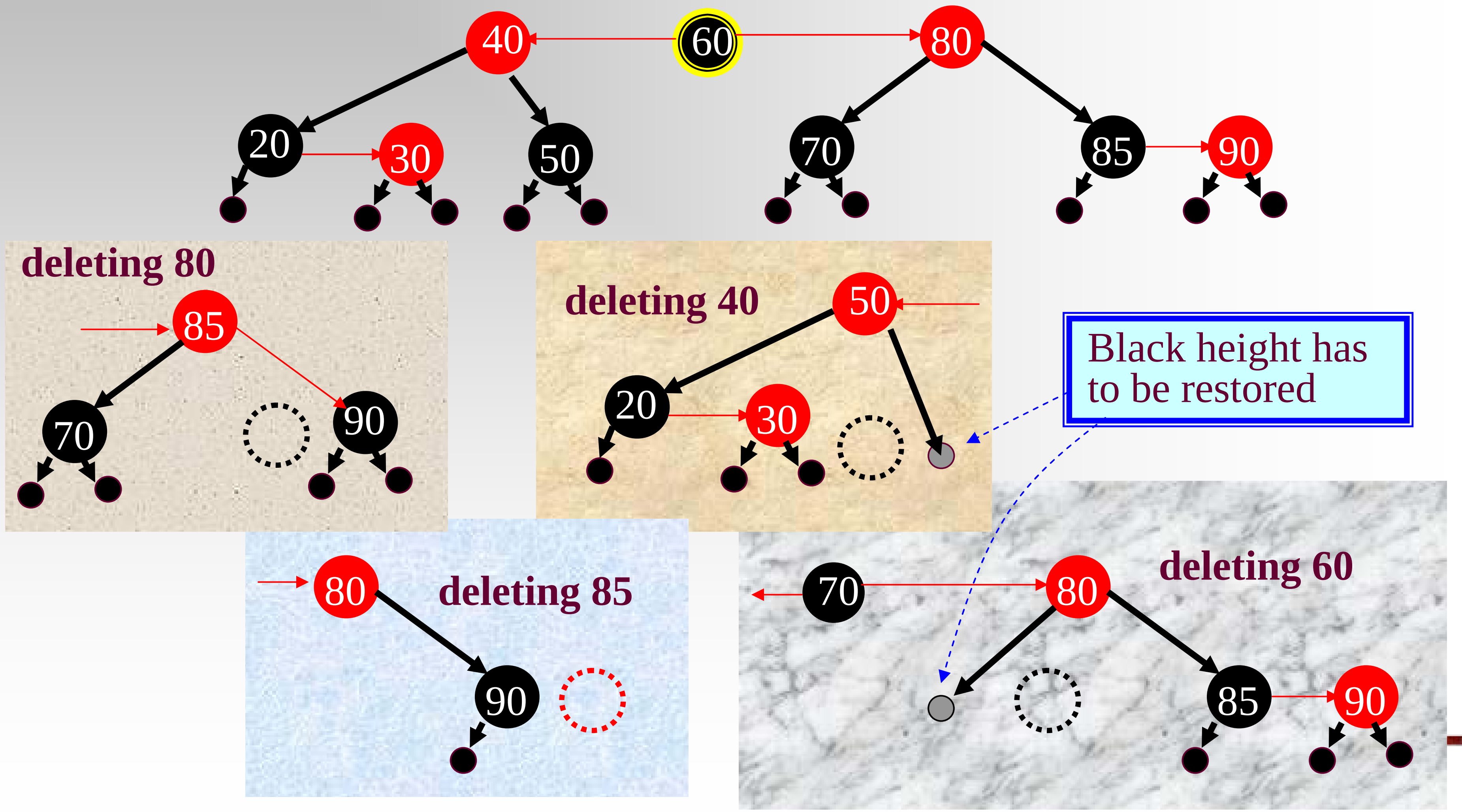
1. Do a standard BST search to locate the node to be logically deleted, call it u

2. If the right child of u is an external node, identify u as the node to be structurally deleted.

3. If the right child of u is an internal node, find the tree successor of u , call it σ, copy the key and information from σ to u. (color of u not changed) Identify σ as the node to be deleted structurally.

4. Carry out the structural deletion and repair any imbalance of black height.

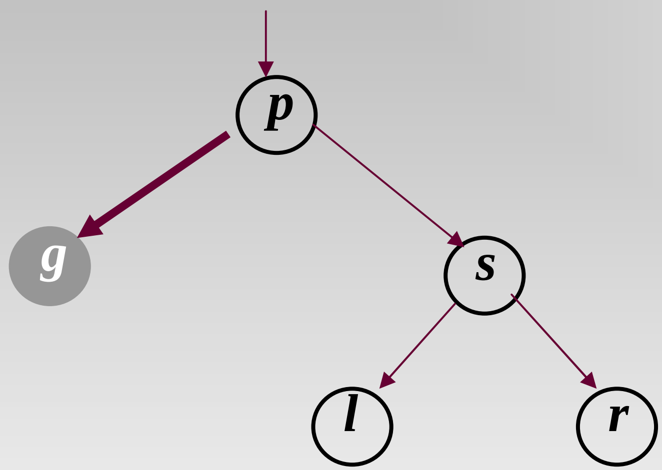
Imbalance of Black Height



Analysis of Black Imbalance

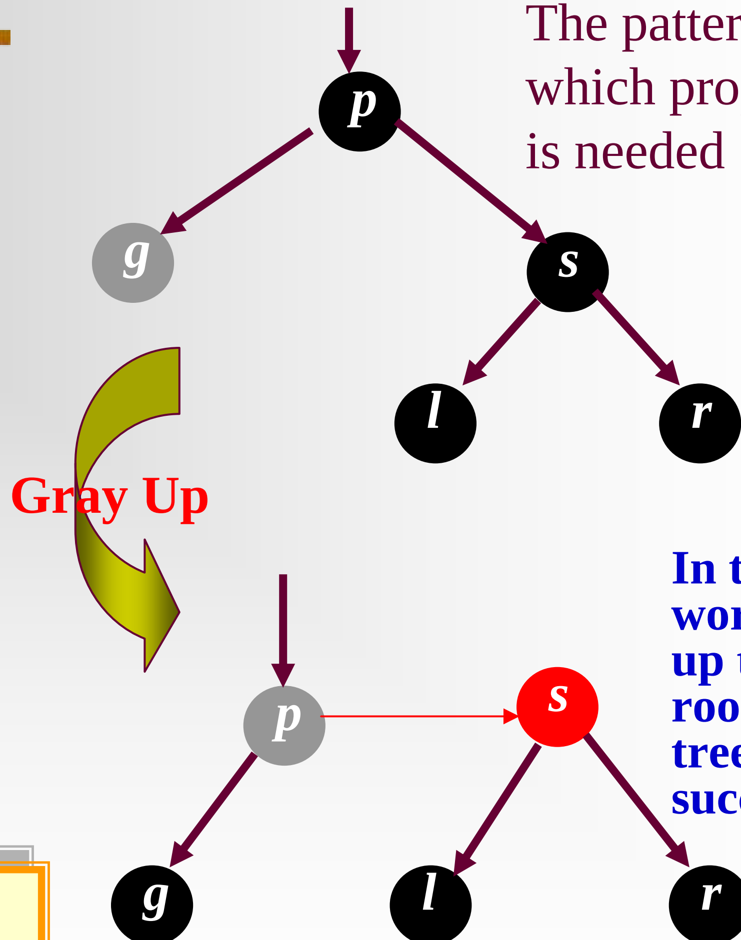
- The imbalance occurs when:
  - A black node is delete structurally, and
  - Its right subtree is black (external)
- The result is:
  - An $RB_{h\text{-}1}$ occupies the position of an RB_h as required by its parent, coloring it as a “gray” node.
- Solution:
  - Find a red node and turn it black as locally as possible.
  - The gray color might propagate up the tree.

Propagation of Gray Node



Map of the vicinity of ***g***, the gray node

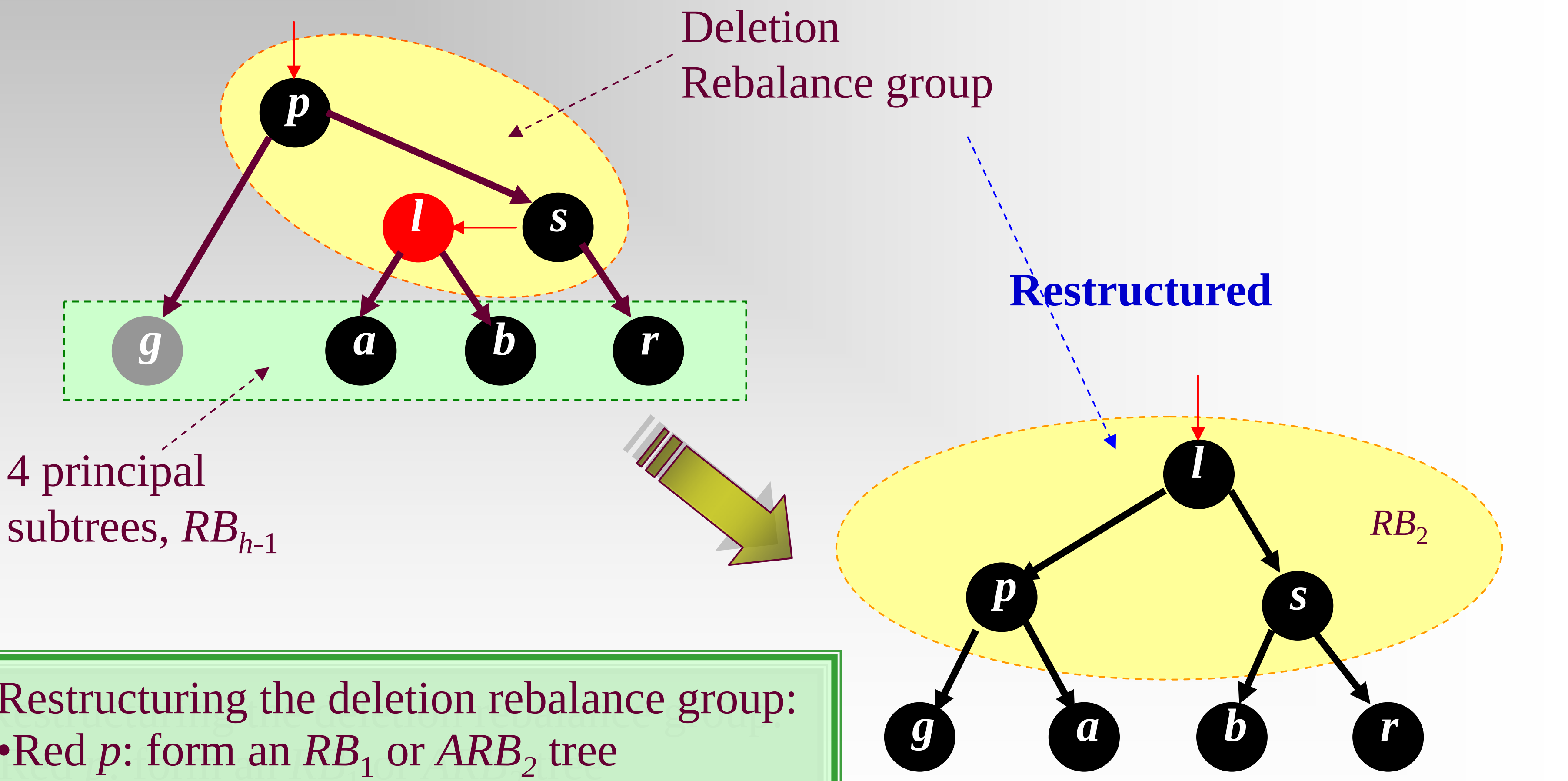
The pattern for which propagation is needed



In the worst case, up to the root of the tree, and successful

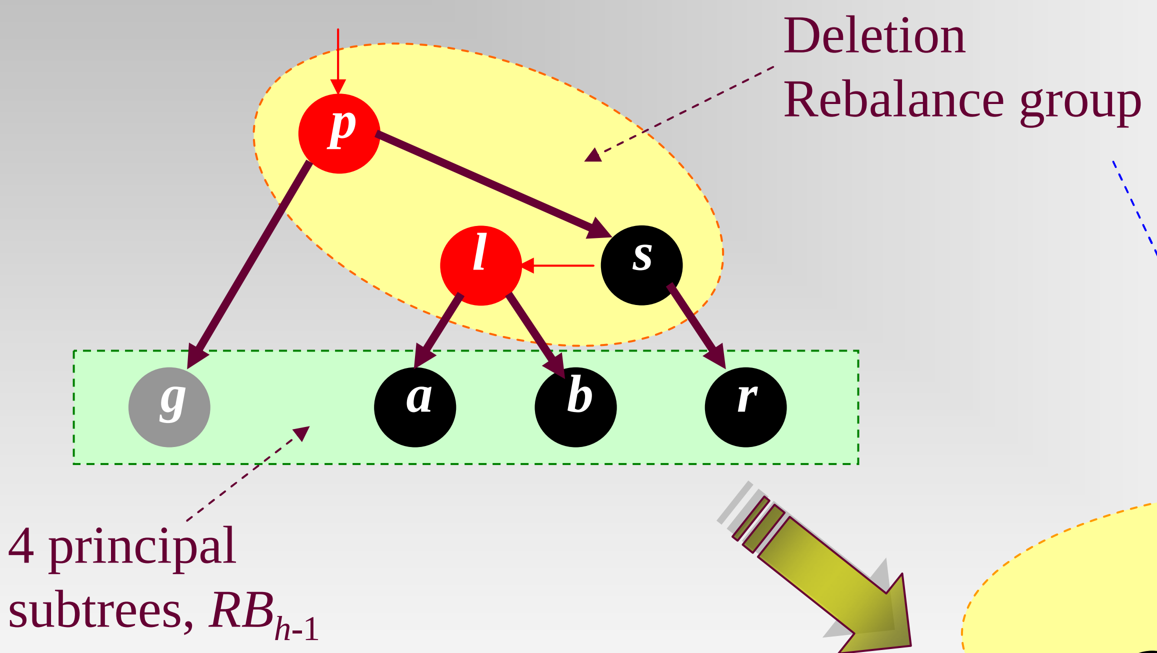
g-subtree gets well-defined black height, but that is less than that required by its parent

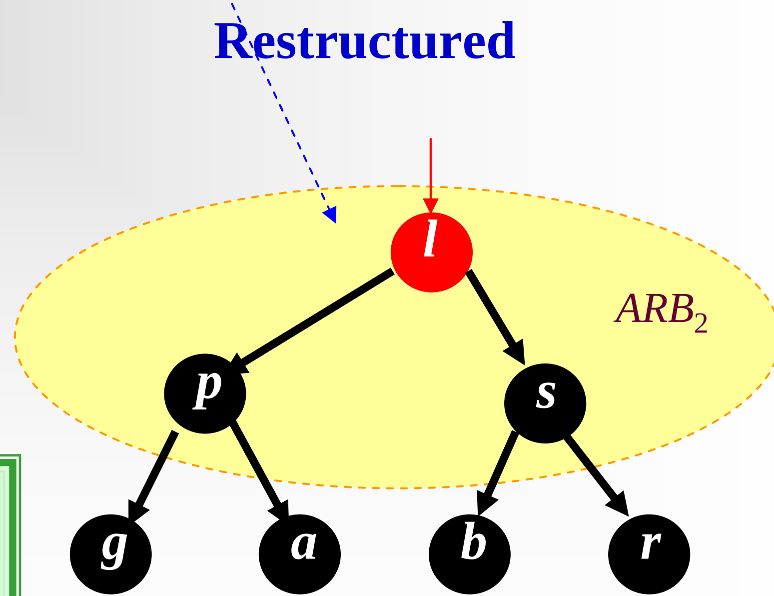
Repairing without Propagation



Restructuring the deletion rebalance group:
- Red *p*: form an RB_1 or ARB_2 tree
- Black *p*: form an RB_2 tree

Repairing without Propagation





Restructuring the deletion rebalance group:
- Red *p*: form an RB_1 or ARB_2 tree
- Black *p*: form an RB_2 tree

Complexity of Operations on RBTree

- With reasonable implementation
  - A new node can be inserted correctly in a red-black tree with n nodes in $\Theta(\log n)$ time in the worst case.
  - Repairs for deletion do $O(1)$ structural changes, but may do $O(\log n)$ color changes.

Home Assignments

- pp.302-
  - 6.4-6
  - 6.11-13
  - 6.17