# Hashing

Algorithm : Design & Analysis

[09]

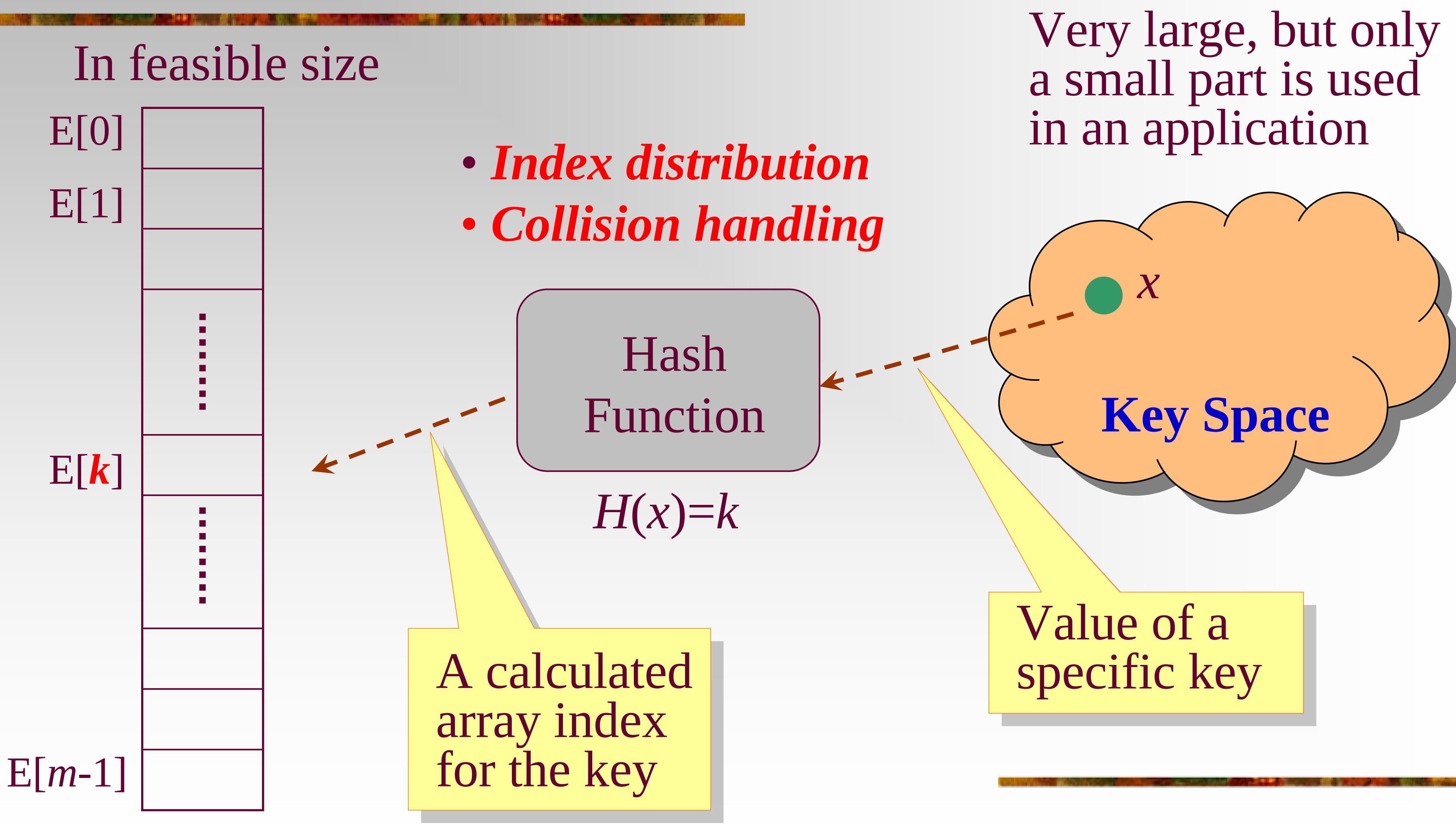
# In the last class…

- Implementing Dictionary ADT
- Definition of red-black tree
- Black height
- Insertion into a red-black tree
- Deletion from a red-black tree

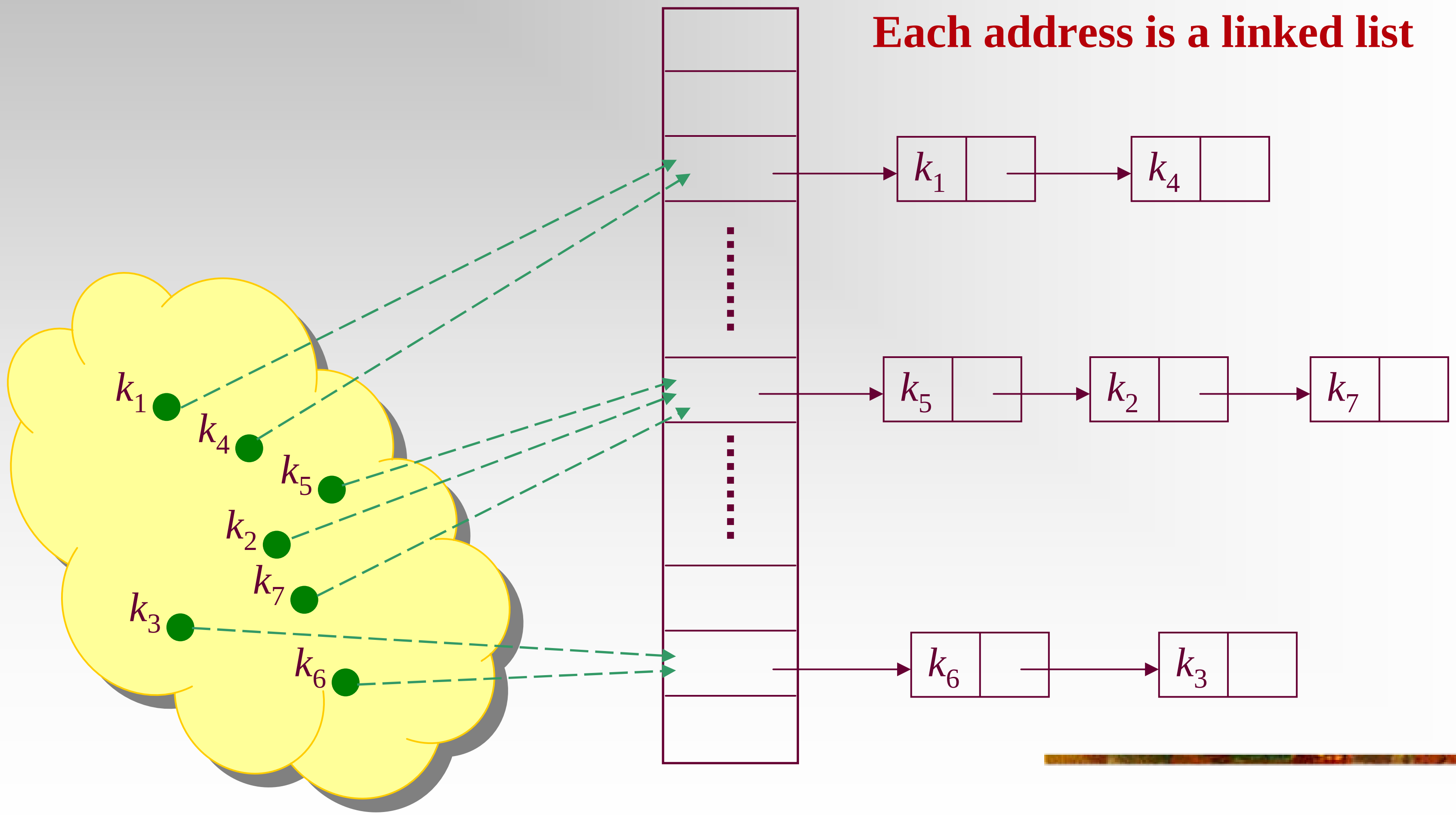
# Hashing

- Hashing
- Collision Handling for Hashing
  - Closed Address Hashing
  - Open Address Hashing
- Hash Functions
- Array Doubling and Amortized Analysis

# Hashing: the Idea



# Collision Handling: Closed Address



# Closed Address: Analysis

- Assumption: simple uniform hashing: for $j=0,1,2,\ldots,m-1$, the average length of the list at $E[j]$ is $n/m$.
- The average cost of an unsuccessful search:
  - Any key that is not in the table is equally likely to hash to any of the $m$ address. The average cost to determine that the key is not in the list $E[h(k)]$ is the cost to search to the end of the list, which is $n/m$. So, the total cost is $\Theta(1+ n/m)$.

# Closed Address: Analysis(cont.)

- For successful search: (assuming that $x_i$ is the $i$th element inserted into the table, $i$=1,2,...,$n$)
  - For each $i$, the probability of that $x_i$ is searched is 1/$n$.
  - For a specific $x_i$, the number of elements examined in a successful search is $t$+1, where $t$ is the number of elements iserted into the same list as $x_i$, after $x_i$ has been inserted. And for any $j$, the probability of that $x_j$ is inserted into the same list of $x_i$ is 1/$m$. So, the cost is:

$$\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right)$$

Cost for computing hashing

Expected number of elements in front of the searched one in the same linked list.

# Closed Address: Analysis(cont.)

- The average cost of a successful search:
  - Define α=n/m as ***load factor***,

The average cost of a successful search is :

$$\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right)=1+\frac{1}{nm}\sum_{i=1}^{n}(n-i)=1+\frac{1}{nm}\sum_{i=1}^{n-1}i$$

$$=1+\frac{n-1}{2m}=1+\frac{\alpha}{2}-\frac{\alpha}{2n}=\Theta(1+\alpha)$$

Number of elements in front of the searched one in the same linked list.

Cost for computing hashing

# Collision Handling: Open Address

- All elements are stored in the hash table, no linked list is used. So, $\alpha$, the load factor, can not be larger than 1.
- Collision is settled by "rehashing": a function is used to get a new hashing address for each collided address, i.e. the hash table slots are *probed* successively, until a valid location is found.
- The probe sequence can be seen as a permutation of $(0,1,2,\ldots, m\text{-}1)$

# Commonly Used Probing

Linear probing:

Given an ordinary hash function $h'$, which is called an auxiliary hash function, the hash function is: **(clustering may occur)**

$$h(k,i) = (h'(k)+i) \bmod m \quad (i=0,1,\ldots,m-1)$$

Quadratic Probing:

Given auxiliary function $h'$ and nonzero auxiliary constant $c_1$ and $c_2$, the hash function is: **(secondary clustering may occur)**

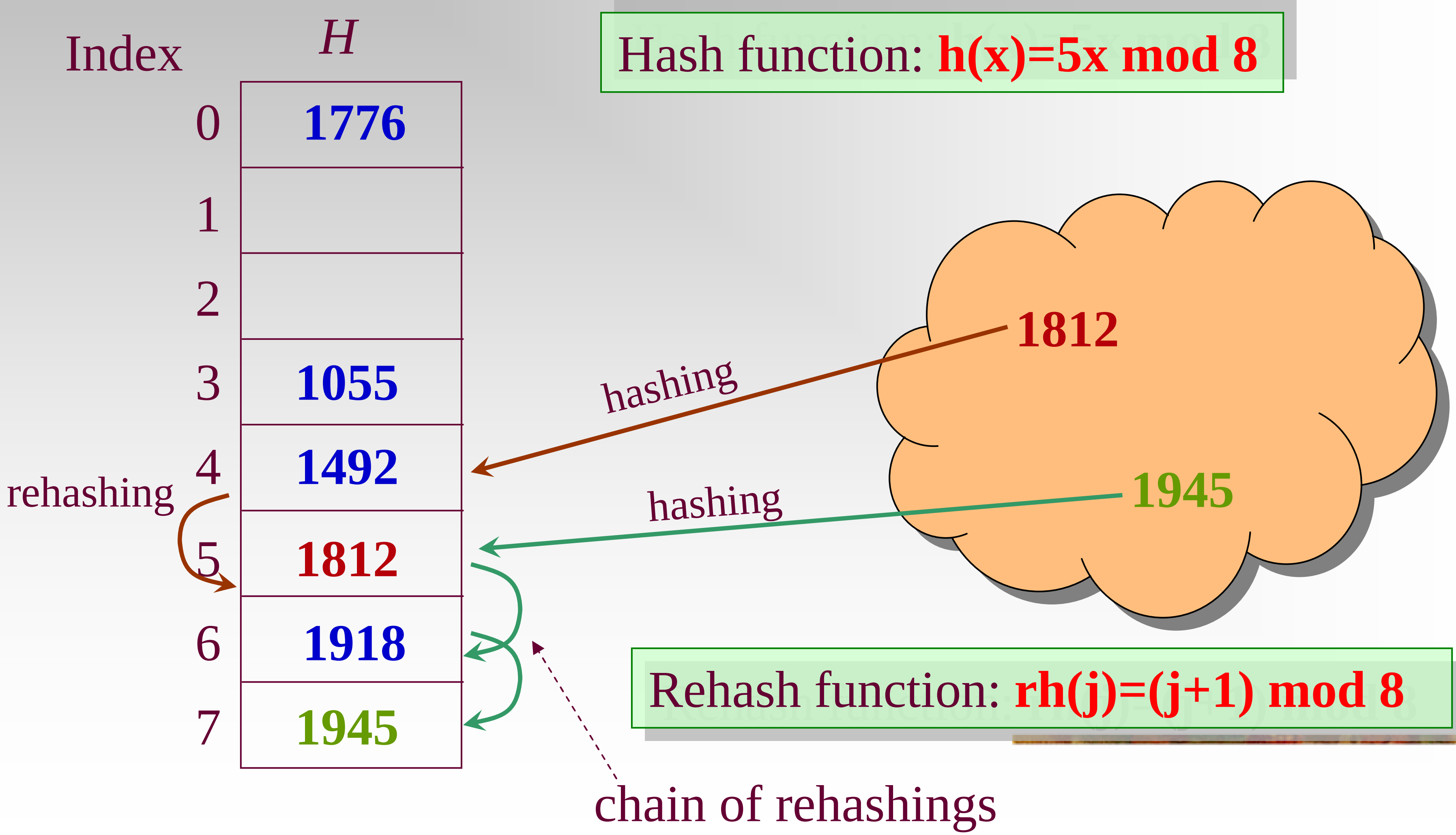
$$h(k,i) = (h'(k)+c_1 i+ c_2 i^2) \bmod m \quad (i=0,1,\ldots,m-1)$$

Double hashing:

Given auxiliary functions $h_1$ and $h_2$, the hash function is:

$$h(k,i) = (h_1(k)+ ih_2(k)) \bmod m \quad (i=0,1,\ldots,m-1)$$

# Linear Probing: an Example



# Equally Likely Permutations

- Assumption: each key is equally likely to have any of the $m$! permutations of (1,2...,$m$-1) as its probe sequence.
- Note: both linear and quadratic probing have only $m$ distinct probe sequence, as determined by the first probe.

# Analysis for Open Address Hash

- Assuming uniform hashing, the average number of probes in an unsuccessful search is at most 1/(1-α) (α=*n*/*m*<1)

Note : the probability of the first probed position being occupied is $\frac{n}{m}$, and that of the $j$th( $j > 1$) position occupied is $\frac{n\text{-}j+1}{m\text{-}j+1}$,

so, the probability of the number of probe no less than $i$ will be :

$$\frac{n}{m}\cdot\frac{n-1}{m-1}\cdot\frac{n-2}{m-2}\cdot\ \cdots\ \cdot\frac{n-i+2}{m-i+2}\le\left(\frac{n}{m}\right)^{i-1}=\alpha^{i-1}$$

Then, the average number of probe is : $\sum_{i=1}^{\infty}\alpha^{i-1}=\sum_{i=0}^{\infty}\alpha^{i}=\frac{1}{1-\alpha}$

# Analysis for Open Address Hash

- Assuming uniform hashing, the average cost of probes in an successful search is at most $\frac{1}{\alpha}\ln\frac{1}{1-\alpha}$ ($\alpha = n/m < 1$)

To search for the $(i+1)$th inserted element in the table, the cost is the same as the cost for inserting it when there are just $i$ elements in the table. At that time, $\alpha = \frac{i}{m}$, so,

the cost is $\frac{1}{1-\frac{i}{m}} = \frac{m}{m-i}$

For your reference:

Half full: 1.387; 90% full: 2.559

So, the cost is :

$$\frac{1}{n}\sum_{i=0}^{n-1}\frac{m}{m-i} = \frac{m}{n}\sum_{i=0}^{n-1}\frac{1}{m-i} = \frac{1}{\alpha}\sum_{i=m-n+1}^{m}\frac{1}{i} \le \frac{1}{\alpha}\int_{m-n}^{m}\frac{dx}{x} = \frac{1}{\alpha}\ln\frac{m}{m-n} = \frac{1}{\alpha}\ln\frac{1}{1-\alpha}$$

# Hashing Function

- A good hash function satisfies the assumption of simple uniform hashing.
- Heuristic hashing functions
  - The division method: $h(k)=k \bmod m$
  - The multiplication method: $h(k)=\lfloor m(kA \bmod 1)\rfloor$ $(0<A<1)$
- No single function can avoid the worst case $\Theta(n)$, so, "Universal hashing" is proposed.

- Rich resource about hashing function:
  Gonnet and Baeza-Yates: *Handbook of Algorithms and Data Structures*, Addison-Wesley, 1991

# Array Doubling

- Cost for search in a hash table is $\Theta(1+\alpha)$, then if we can keep $\alpha$ constant, the cost will be $\Theta(1)$
- Space allocation techniques such as array doubling may be needed.
- The problem of "unusually expensive" individual operation.

# Looking at the Memory Allocation

- hashingInsert(HASHTABLE *H*, ITEM *x*)
- **integer** *size*=0, *num*=0;
- **if** *size*=0 **then** allocate a block of size 1; *size*=1;
- **if** *num*=*size* **then**
- allocate a block of size 2*size*;
- move all item into new table;
- *size*=2*size*;
- insert *x* into the table;
- *num*=*num*+1;
- **return**

Insertion with expansion: cost *size*

Elementary insertion: cost 1

# Worst-case Analysis of the Insertion

- For $n$ execution of insertion operations
  - A bad analysis: the worst case for one insertion is the case when expansion is required, up to $n$
  - So, the worst case cost is in $O(n^2)$.
- Note the expansion is required during the $i$th operation only if $i=2^k$, and the cost of the $i$th operation

Of course NOT!

$$c_i = \begin{cases} i & \text{if } i-1 \text{ is exactly power of } 2 \\ 1 & \text{otherwise} \end{cases}$$

$$\text{So, the total cost is: } \sum_{i=1}^{n} c_i \le n + \sum_{j=0}^{\lfloor \lg n \rfloor} 2^j < n + 2n = 3n$$

# Amortized Time Analysis

- Amortized equation:

  *amortized cost = actual cost + accounting cost*

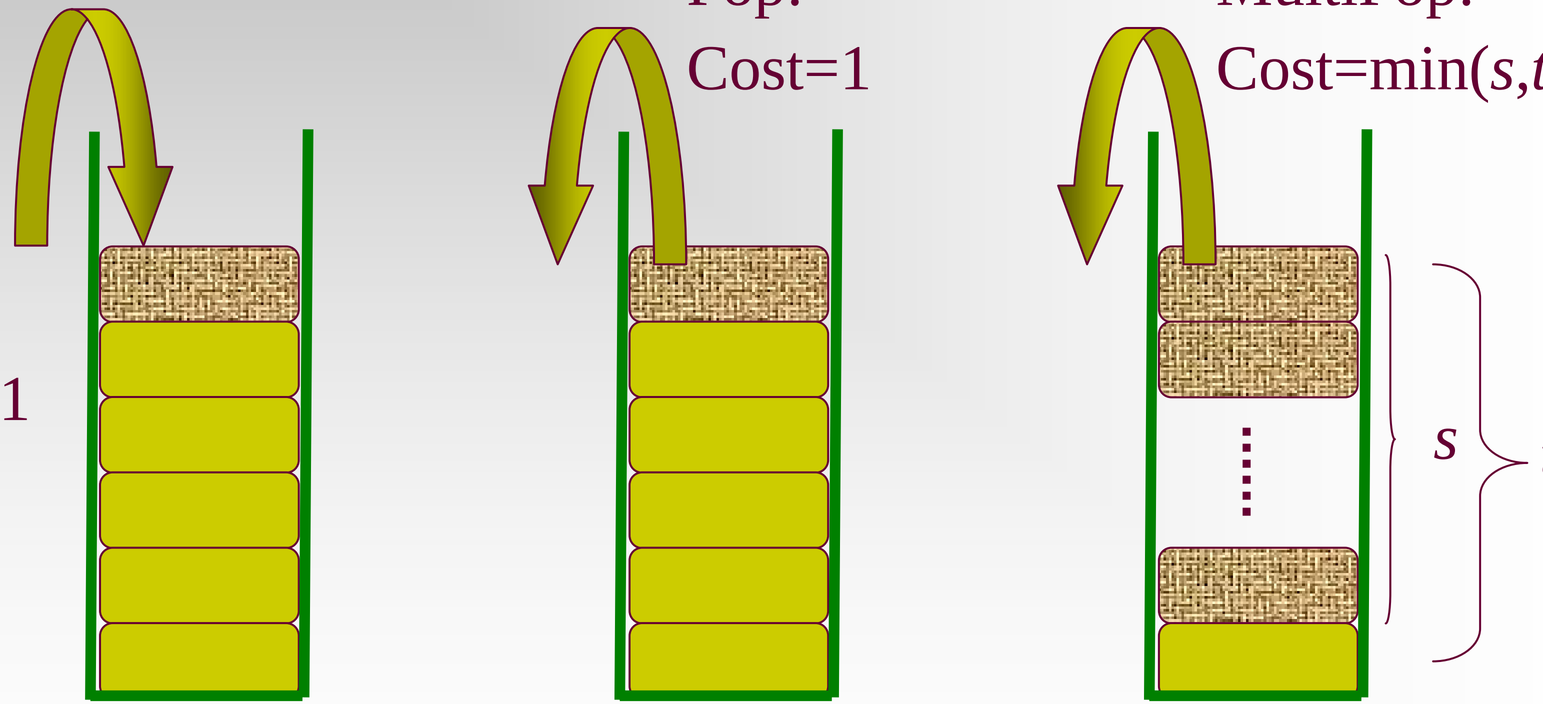
- Design goals for accounting cost
  - In any legal sequence of operations, the sum of the accounting costs is nonnegative.
  - The amortized cost of each operation is fairly regular, in spite of the wide fluctuate possible for the actual cost of individual operations.

# Amortized Analysis: MultiPop Stack

Push:
Cost=1

Pop:
Cost=1

MultiPop:
Cost=min($s$,$t$)




Amortized cost: push:2; pop, multipop: 0

# Amortized Analysis: Binary Counter

| | | |
|---|---|---|
| 0 | 0 0 0 0 0 0 0 0 | 0 |
| 1 | 0 0 0 0 0 0 0 1 | 1 |
| 2 | 0 0 0 0 0 0 1 0 | 3 |
| 3 | 0 0 0 0 0 0 1 1 | 4 |
| 4 | 0 0 0 0 0 1 0 0 | 7 |
| 5 | 0 0 0 0 0 1 0 1 | 8 |
| 6 | 0 0 0 0 0 1 1 0 | 10 |
| 7 | 0 0 0 0 0 1 1 1 | 11 |
| 8 | 0 0 0 0 1 0 0 0 | 15 |
| 9 | 0 0 0 0 1 0 0 1 | 16 |
| 10 | 0 0 0 0 1 0 1 0 | 18 |
| 11 | 0 0 0 0 1 0 1 1 | 19 |
| 12 | 0 0 0 0 1 1 0 0 | 22 |
| 13 | 0 0 0 0 1 1 0 1 | 23 |
| 14 | 0 0 0 0 1 1 1 0 | 25 |
| 15 | 0 0 0 0 1 1 1 1 | 26 |
| 16 | 0 0 0 1 0 0 0 0 | 31 |

Cost measure: bit flip

amortized cost:

set 1: 2

set 0: 0

# Accounting Scheme for Stack Push

- Push operation with array doubling
  - No resize triggered: 1
  - Resize($n \rightarrow 2n$) triggered: $tn+1$ (t is a constant)
- Accounting scheme (specifying accounting cost)
  - No resize triggered: $2t$
  - Resize($n \rightarrow 2n$) triggered: $-nt+2t$
- So, the amortized cost of each individual push operation is **$1+2t \in \Theta(1)$**

# Home Assignment

- pp.302-
  - 6.1
  - 6.2
  - 6.18
  - 6.19