Graph Traversals

Algorithm : Design & Analysis

[11]

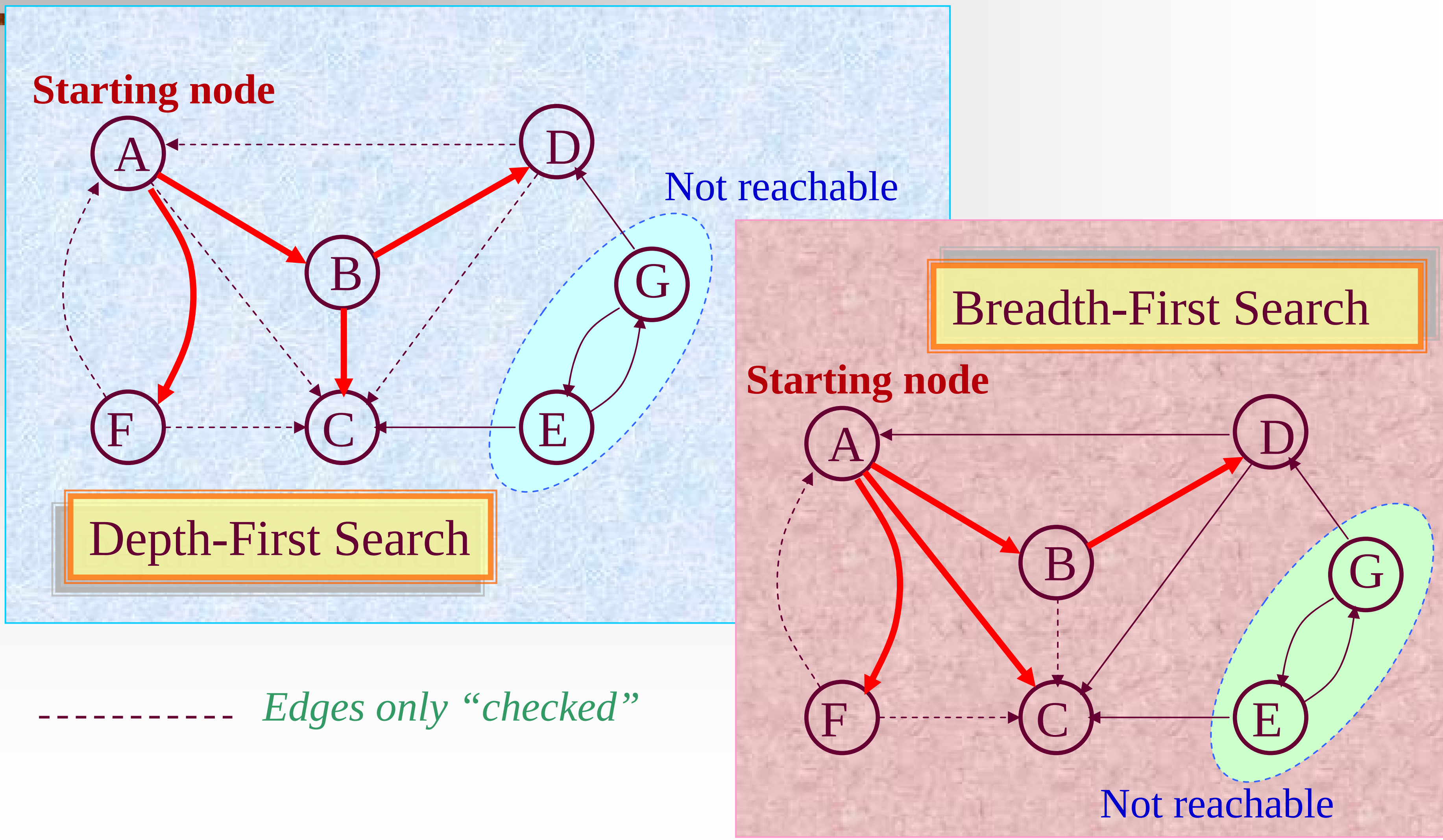
In the last class…

- Dynamic Equivalence Relation
- Implementing Dynamic Set by Union-Find
  - Straight Union-Find
  - Making Shorter Tree by Weighted Union
  - Compressing Path by Compressing-Find
- Amortized Analysis of *wUnion-cFind*

Graph Traversals

- Depth-First and Breadth-First Search
- Finding Connected Components
- General Depth-First Search Skeleton
- Depth-First Search Trace

Graph Traversal: an Example



Outline of Depth-First Search

- dfs(G,v)
- Mark v as “discovered”.
- For each vertex w that edge vw is in G:
- If w is undiscovered:
- dfs(G,w)
- Otherwise:
- “Check” vw without visiting w.
- Mark v as “finished”.

A vertex must be exact one of three different status:

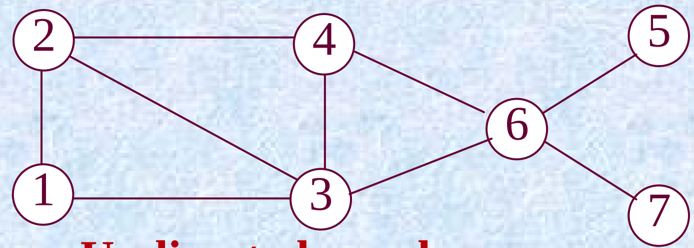
- undiscovered
- discovered but not finished
- finished

That is: exploring vw, visiting w, exploring from there as much as possible, and backtrack from w to v.

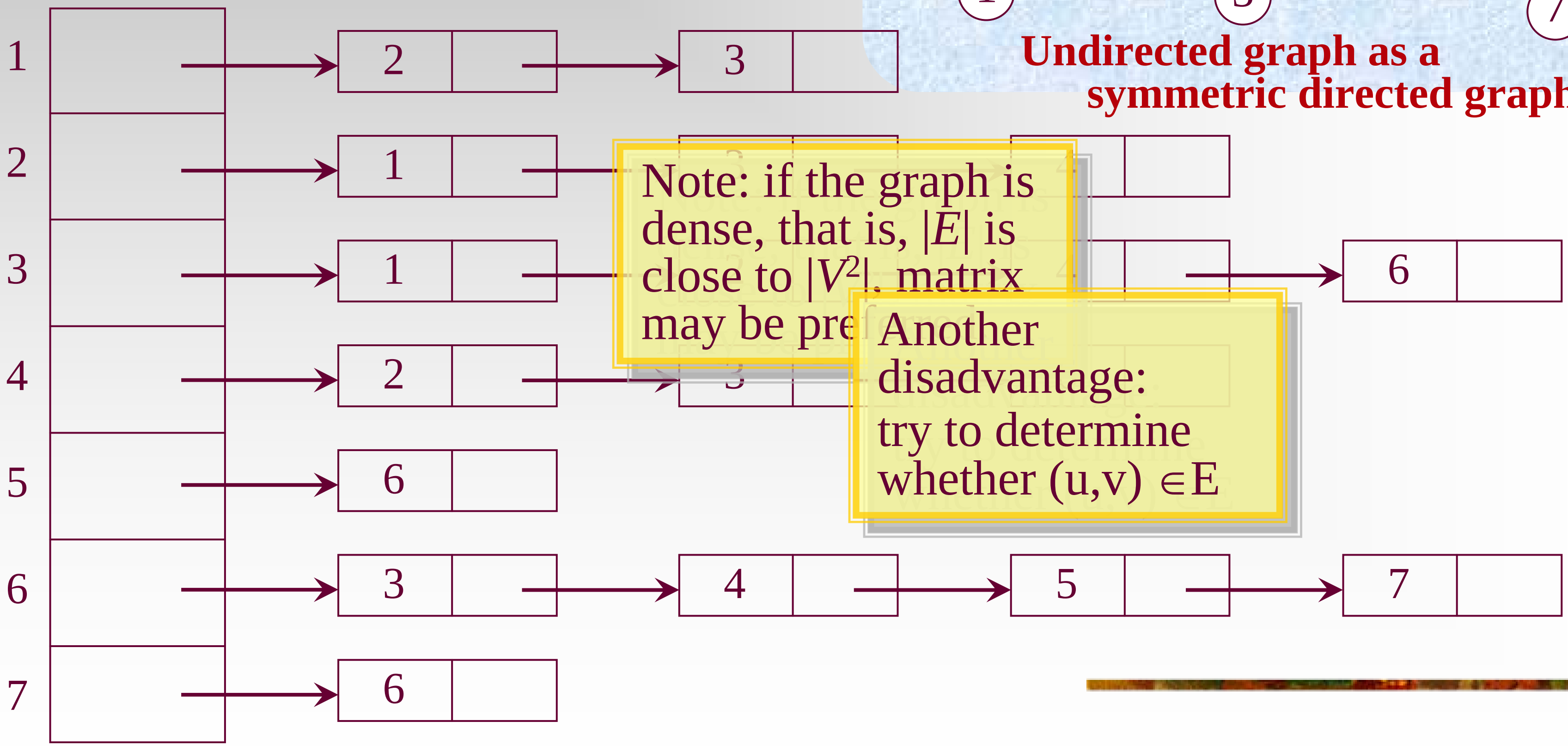
Outline of Breadth First Search

- Bfs(G,*s*)
-     Mark *s* as “discovered”;
-     enqueue(pending,*s*);
-     **while** (pending is nonempty)
-         dequeue(pending, *v*);
-         For each vertex *w* that edge *vw* is in G:
-             If *w* is “undiscovered”
-                 Mark *w* as “discovered” and enqueue(pending, *w*)
-         Mark *v* as “finished”;

Graph as Group of Linked-List



Undirected graph as a symmetric directed graph



Note: if the graph is dense, that is, $|E|$ is close to $|V^2|$, matrix may be preferred.

Another disadvantage: try to determine whether $(u,v) \in E$

Finding Connected Components

- Input: a symmetric digraph G, with *n* nodes and 2*m* edges(interpreted as an undirected graph), implemented as a array *adjVertices*[1,…*n*] of adjacency lists.
- Output: an array *cc*[1..*n*] of component number for each node v_i

```
void connectedComponents(Intlist[ ] adjVertices, int n,
      int[ ] cc) // This is a wrapper procedure
   int[ ] color=new int[n+1];
   int v;
   <Initialize color array to white for all vertices>
   for (v=1; v≤n; v++)
      if (color[v]==white)
         ccDFS(adjVertices, color, v, v, cc);
   return
```

Depth-first search

ccDFS: the procedure

- **void** ccDFS(IntList[ ] *adjVertices*, **int**[ ] *color*, **int** *v*, **int** *ccNum*, **int** [ ] *cc*)//*v* as the code of current connected component
- **int** *w*;
- IntList remAdj;
- color[v]=gray;
- cc[v]=ccNum;
- remAdj=adjVertices[v];
- **while** (remAdj≠nil)
- w=first(remAdj);
- **if** (color[w]==white)
- ccDFS(adjVertices, color, w, ccNum, cc);
- remAdj=rest(remAdj);
- color[v]=black;
- **return**

The elements of *remAdj* are neighbors of *v*

Processing the next neighbor, if existing, another depth-first search to be incurred

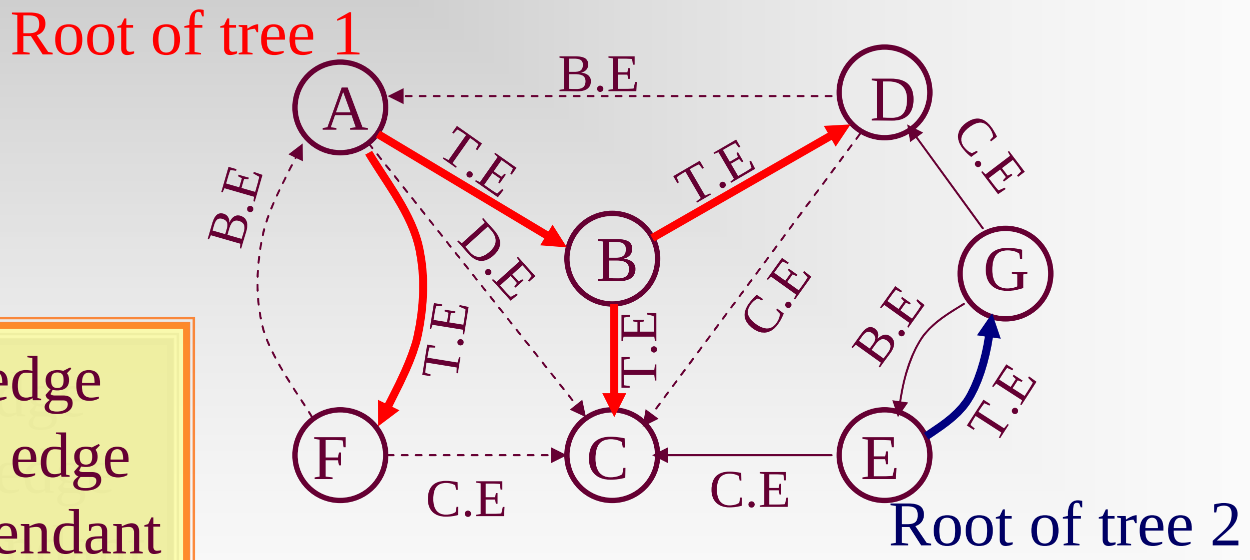
v finished

Analysis of CC Algorithm

- connectedComponents, the wrapper
  - Linear in *n* (color array initialization+for loop on *adjVertices*)
- ccDFS, the depth-first searcher
  - In one execution of ccDFS on *v*, the number of instructions(rest(*remAdj*)) executed is proportional to the size of *adjVertices*[*v*].
  - Note: Σ(size of *adjVertices*[*v*]) is 2*m*, and the adjacency lists are traveresed **only once**.
- So, the complexity is in Θ(m+n)
- Extra space requirements:
  - color array
  - activation frame stack for recursion

Depth-First Search Trees

DFS forest={(DFS tree1), (DFS tree2)}



T.E: tree edge
B.E: back edge
D.E: descendant edge
C.E: cross edge

A finished vertex is never revisited, such as C

Visits On a Vertex

- Classification for the visits on a vertex
  - First visit(exploring): status: white$\rightarrow$gray
  - (Possibly) multi-visits by backtracking to: status keeps gray
  - Last visit(no more branch-finished): status: gray$\rightarrow$black
- Different operations can be done on the vertex or (selected) incident edges during the different visits on a specific vertex

Depth-First Search: Generalized

- Input: Array *adjVertices* for graph G
- Output: Return value depends on application.
- **int** dfsSweep(IntList[] *adjVertices*,**int** n, …)
- **int** ans;
- <Allocate color array and initialize to white>
- For each vertex *v* of G, in some order
- **if** (color[v]==white)
- **int** vAns=**dfs(*adjVertices*, color, v, …);**
- <Process vAns>
- // Continue loop
- **return** ans;

Depth-First Search: Generalized

```
int dfs(IntList[] adjVertices, int[] color, int v, …)
    int w;
    IntList remAdj;
    int ans;
    color[v]=gray;
    <Preorder processing of vertex v>
    remAdj=adjVertices[v];
    while (remAdj≠nil)
        w=first(remAdj);
        if (color[w]==white)
            <Exploratory processing for tree edge vw>
            int wAns=dfs(adjVertices, color, w, …);
            < Backtrack processing for tree edge vw , using wAns>
        else
            <Checking for nontree edge vw>
        remAdj=rest(remAdj);
    <Postorder processing of vertex v, including final computation of ans>
    color[v]=black;
    return ans;
```

If partial search is used for a application, tests for termination may be inserted here.

Specialized for connected components:
- parameter added
- preorder processing inserted – cc[v]=ccNum

Breadth-First Search: the Skeleton

- Input: Array *adjVertices* for graph G
- Output: Return value depends on application.
- **void** bfsSweep(IntList[] *adjVertices*,**int** n, …)
-     **int** ans;
-     <Allocate color array and initialize to white>
-     For each vertex *v* of G, in some order
-         **if** (color[v]==white)
-             **void bfs(*adjVertices*, color, v, …);**
-     // Continue loop
-     **return**;

Breadth-First Search: the Skeleton

```
void bfs(IntList[] adjVertices, int[] color, int v, …)
    int w; IntList remAdj; Queue pending;
    color[v]=gray;     enqueue(pending, v);
    while (pending is nonempty)
        w=dequeue(pending); remAdj=adjVertices[w];
        while (remAdj≠nil)
            x=first(remAdj);
            if (color[x]==white)
                color[x]=gray; enqueue(pending, x);
                remAdj=rest(remAdj);
        <processing of vertex v>
        color[w]=black;
    return ;
```

DFS vs. BFS Search

- Processing Opportunities for a node
  - Depth-first: 2
    - At discovering
    - At finishing
  - Breadth-first: only 1, when de-queued
  - At the second processing opportunity for the DFS, the algorithm can make use of information about the descendants of the current node.

Time Relation on Changing Color

- Keeping the order in which vertices are encountered for the first or last time
  - A global interger time: 0 as the initial value, incremented with each color changing for *any* vertex, and the final value is $2n$
  - Array *discoverTime*: the i th element records the time vertex v_i turns into gray
  - Array *finishTime*: the i th element records the time vertex v_i turns into black
  - The active interval for vertex v, denoted as *active*(v), is the duration while v is gray, that is:

$$discoverTime[v], \ldots, finishTime[v]$$

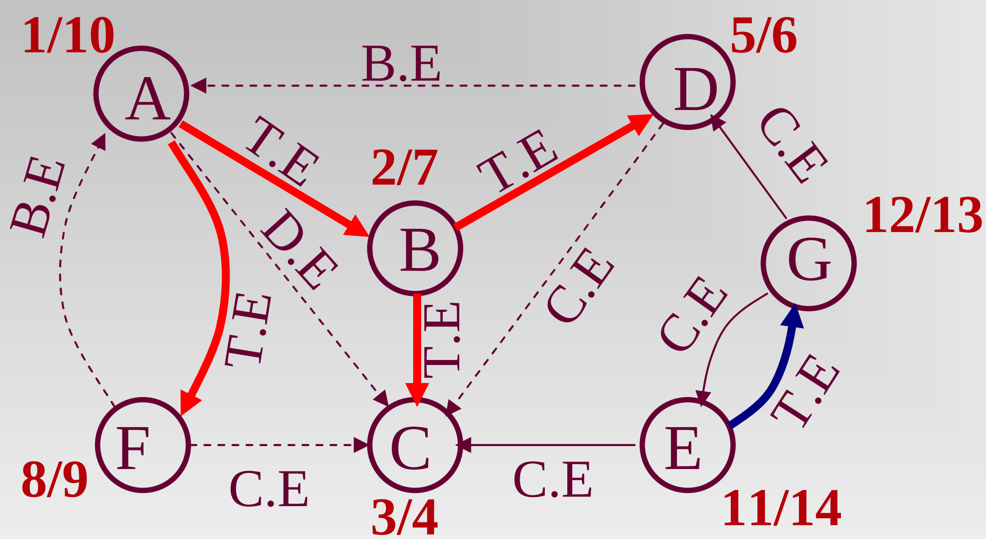
Depth-First Search Trace

- General DFS skeleton modified to compute discovery and finishing times and “construct” the depth-first search forest.
- **int** dfsTraceSweep(IntList[ ] *adjVertices*,**int** n, **int**[ ] *discoverTime*, **int**[ ] *finishTime*, **int**[ ] *parent*)
- **int** ans; **int** *time*=0
- <Allocate color array and initialize to white>
- For each vertex *v* of G, in some order
- **if** (color[v]==white)
- parent[v]=-1
- **int** vAns=dfsTrace(*adnVertices*, color, v, *discoverTime*, *finishTime*, *parent*, *time* **);**
- // Continue loop
- **return** ans;

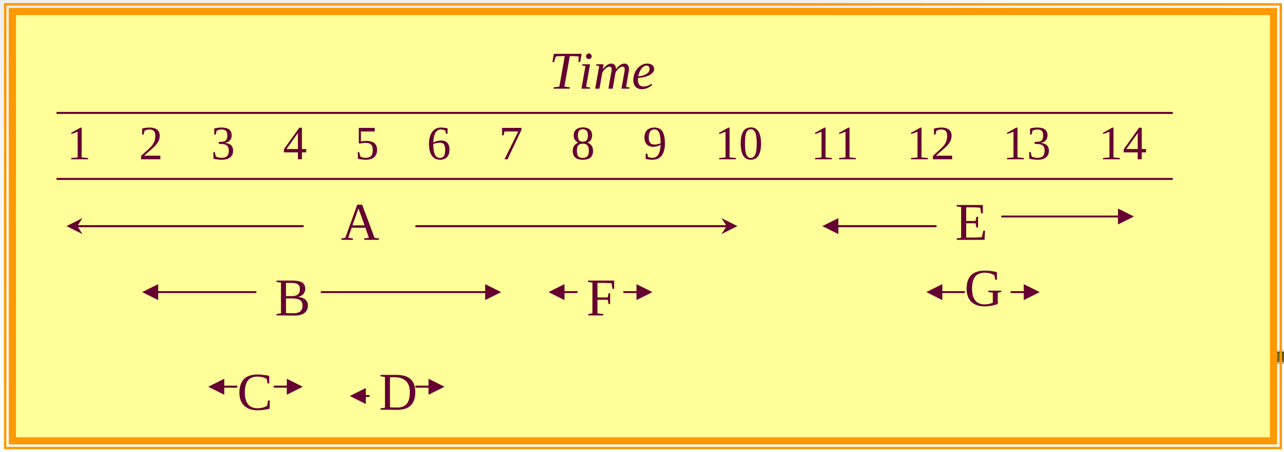
Depth-First Search Trace

```
int dfsTrace(intList[ ] adjVertices, int[ ] color, int v,  int[ ] discoverTime,
                    int[ ] finishTime, int[ ] parent  int time)
    int w; IntList remAdj; int ans;
    color[v]=gray; time++; discoverTime[v]=time;
    remAdj=adjVertices[v];
    while (remAdj≠nil)
        w=first(remAdj);
        if (color[w]==white)
            parent[w]=v;
            int wAns=dfsTrace(adjVertices, color, w, discoverTime, finishTime,
                                        parent, time);
        else  <Checking for nontree edge vw>
        remAdj=rest(remAdj);
     time++; finishTime[v]=time; color[v]=black;
    return ans;
```

Edge Classification and the Active Intervals



The relations are summarized in the next frame



Properties about Active Intervals(1)

- If *w* is a descendant of *v* in the DFS forest, then *active*(*w*)$\subseteq$*active*(*v*), and the inclusion is proper if w$\neq$v.
- Proof:
  - Define a partial order <: w<v iff. w is a proper descendants of v in its DFS tree. The proof is by induction on <)
  - If v is minimal. The only descendant of v is itself. Trivial.
  - Assume that for all x<v, if w is a descendant of x, then active(w)$\subseteq$active(x).
  - Let w is any proper descendant of v in the DFS tree, there must be some x such that vx is a tree edge on the tree path to w, so w is a descendant of x. According to dfsTrace, we have *active*(*x*)$\subset$*active*(*v*), by inductive hypothesis, *active*(*w*)$\subset$*active*(*v*),

Properties about Active Intervals(2)

- If v and w have no ancestor/descendant relationship in the DFS forest, then their **active intervals** are disjoint.
- Proof:
  - If v and w are in different DFS tree, it is triavially true, since the trees are processed one by one.
  - Otherwise, there must be a vertex c, satisfying that there are tree paths c to v, and c to w, without edges in common. Let the leading edges of the two tree path are cy, cz, respectively. According to dfsTrace, *active*(y) and *active*(z) are disjoint.
  - We have *active*(v)$\subseteq$*active*(y), *active*(w)$\subseteq$*active*(z). So, *active*(v) and *active*(w) are disjoint.

Properties about Active Intervals(3)

- If *active*(w)$\subseteq$*active*(v), then w is a descendant of v. And if *active*(w)$\subset$*active*(v), then w is a proper descendant of v.

  That is: w is discovered while v is active.

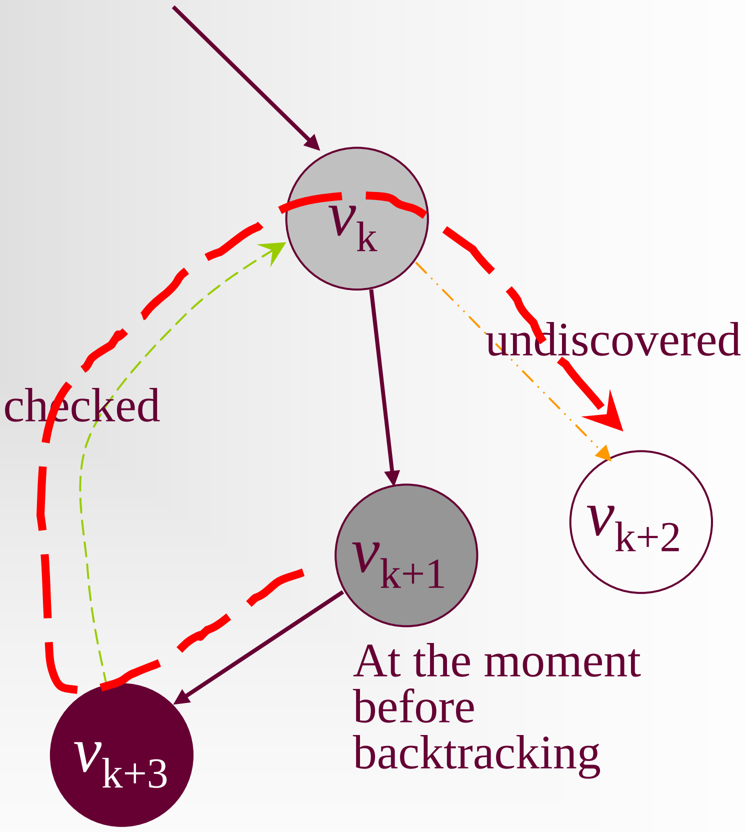
- Proof:
  - If w is **not** a descendant of v, there are two cases:
    - v is a proper descendant of w, then *active*(v)$\subset$*active*(w), so, it is impossible that *active*(w)$\subseteq$*active*(v), contradiction.
    - There is no ancestor/descendant relationship between v and w, then *active*(w) and *active*(v) are disjoint, contradiction.

Properties about Active Intervals(4)

- If edge vw$\in E_G$, then
  - vw is a **cross edge** iff. *active*(w) entirely precedes *active*(v).
  - vw is a **descendant edge** iff. there is some third vertex *x*, such that *active*(w)$\subset$*active*(*x*)$\subset$*active*(*v*),
  - vw is a **tree edge** iff. *active*(w)$\subset$*active*(*v*), and there is no third vertex *x*, such that *active*(w)$\subset$*active*(*x*) $\subset$*active*(*v*),
  - vw is a **back edge** iff. *active*(v)$\subset$*active*(w),

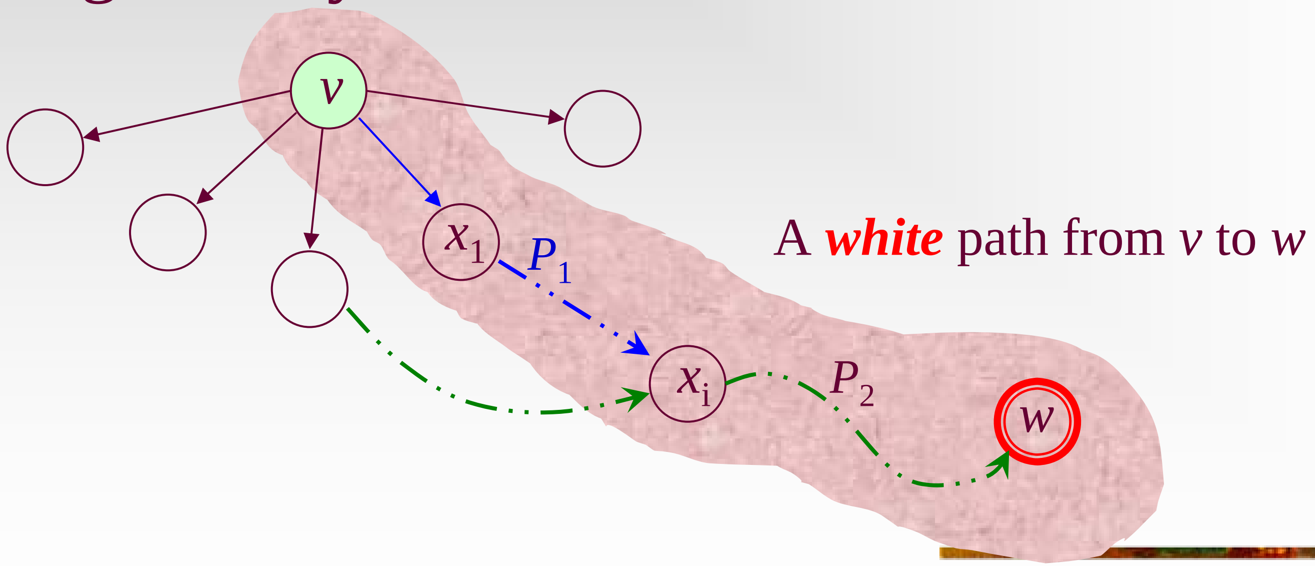
Ancestor/Descendant Relationship and Directed Path

- That *w* is a descendant of *v* in the DFS forest means that there is a direct path from *v* to *w* in some DFS tree.
- The path is also a path in *G*.
- However, if there is a direct path from *v* to *w* **in G**, is *w* necessarily a descendant of *v* in ***the*** DFS forest?



DFS Tree Path

- [White Path Theorem] w is a descendant of v in a DFS tree iff. at the time v is discovered(just to be changing color into gray), there is a path in G from v to w consisting entirely of white vertices.



Proof of White Path Theorem

- Proof
  - $\Rightarrow$ All the vertices in the path are descendants of v.
  - $\Leftarrow$ by induction on the length *k* of a white path from v to w.
    - When k=0, v=w.
    - For k>0, let P=(v, x_1,x_2,…x_k=w). There must be some vertex on P which is discovered during the active interval of v, e.g. x_1, Let x_i is earliest discovered among them. Divide P into P_1 from v to x_i, and P_2 from x_i to w. P_2 is a white path with length less than k, so, by inductive hypothesis, w is a descendant of x_i. Note: $active(x_i) \subseteq active(v)$, so x_i is a descendant of v. By transitivity, w is a descendant of v.

Home Assignments

- 7.12
- 7.14
- 7.15
- 7.16