Applications of Graph Traversal

Algorithm : Design & Analysis

[12]

In the last class…

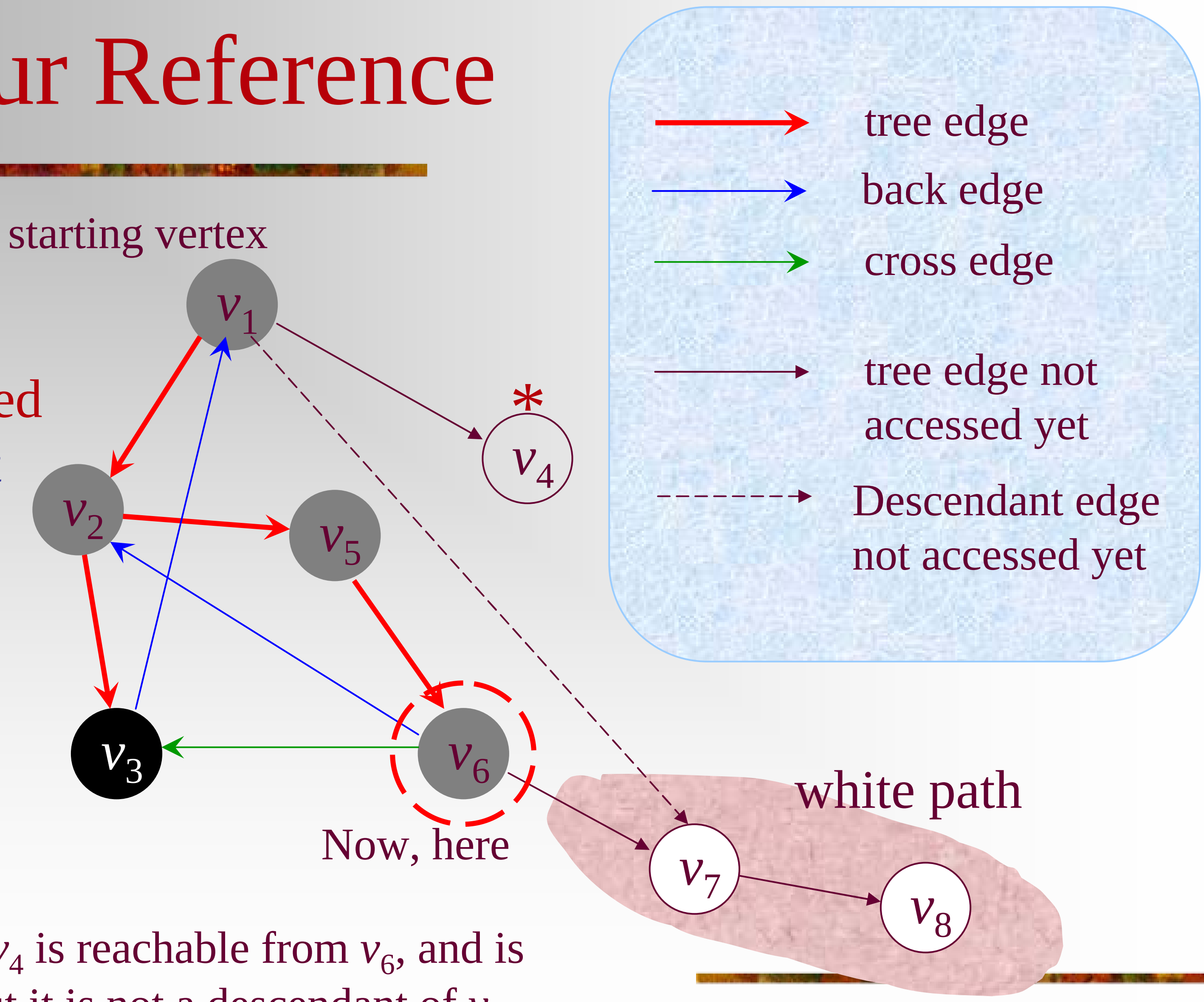
- Depth-First and Breadth-First Search
- Finding Connected Components
- General Depth-First Search Skeleton
- Depth-First Search Trace

Applications of Graph Traversal

- Directed Acyclic Graph
  - Topological Order
  - Critical Path Analysis
- Strongly Connected Component
  - Strong Component and Condensation
  - Leader of Strong Component
  - The Algorithm

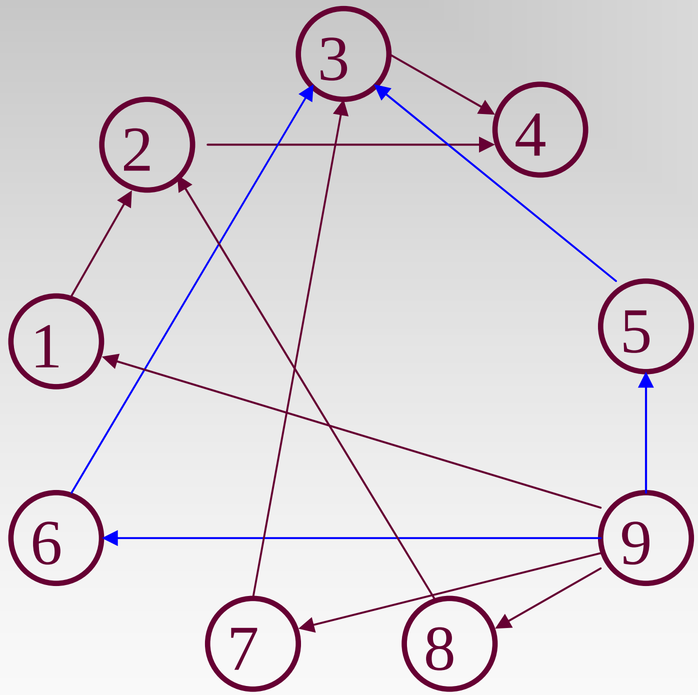
For Your Reference

A DFS tree partially formed at the moment the search checking v_3 from v_6

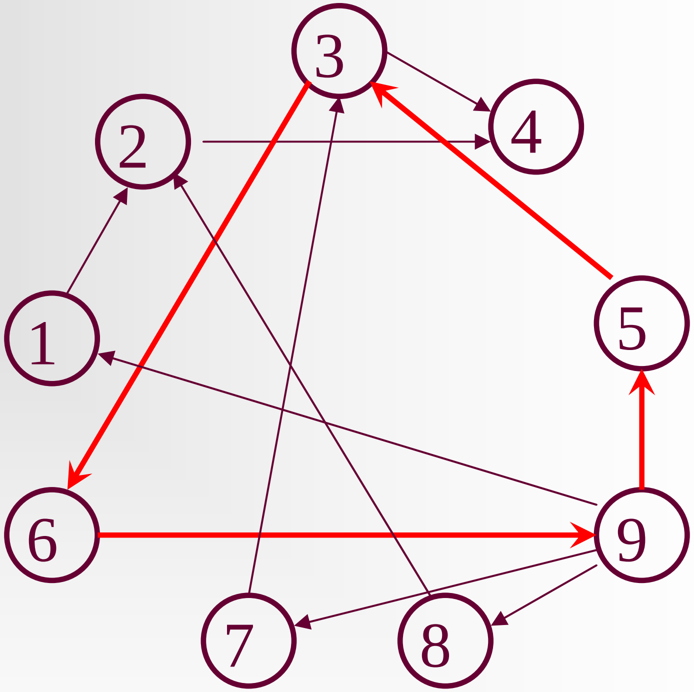


* Note: v_4 is reachable from v_6, and is white, but it is not a descendant of v_6

Directed Acyclic Graph (DAG)



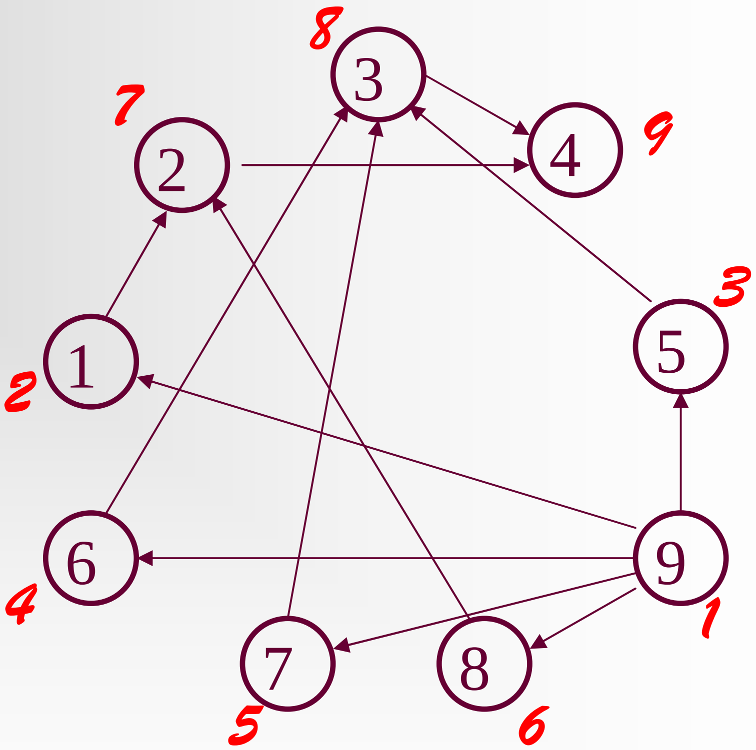
A Directed Acyclic Graph



Not a DAG

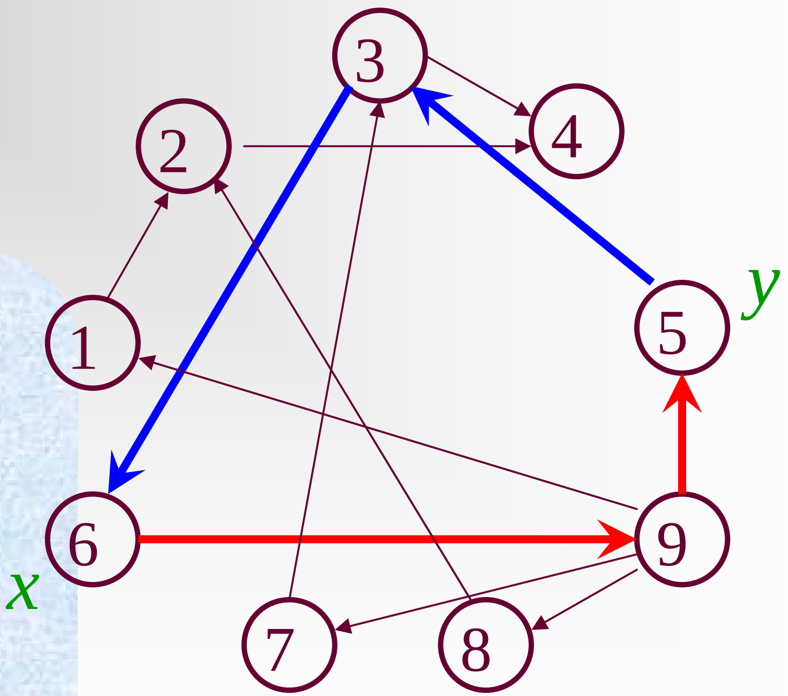
Topological Order

- G=(V,E) is a directed graph with *n* vertices. A **topological order** for G is an assignment of distinct integer 1,2,…, *n* to the vertices of V as their **topological number**, such that, for every *vw*∈E, the topological number of v is less than that of w.
- Reverse topological order can be defined similarly, ("greater than")



Existence of Topological Order
- a Negative Result

- If a directed graph G has a cycle, then G has no topological order
- Proof
  - [By contradiction]



yx-path

xy-path

For any given topological order, all the vertices on both paths must be in increasing order. Contradiction results for any assignments for *x* and *y*.

Reverse Topological Ordering using DFS Skeleton - Parameters

- Specialized parameters
  - Array *topo*, keeps the topological number assigned to each vertex.
  - Counter *topoNum* to provide the integer to be used for topological number assignments
- Output
  - Array *topo* as filled.

Reverse Topological Ordering using DFS Skeleton - Wrapper

- **void** dfsTopoSweep(IntList[ ] *adjVertices*,**int** n, **int**[ ] *topo*)
-     **int** *topoNum*=0
-     <Allocate color array and initialize to white>
-     For each vertex *v* of G, in some order
-         **if** (color[v]==white)
-             **dfsTopo(*adjVertices*, color, v, topo, topoNum);**
-         // *Continue loop*
-     **return**;

For non-reverse topological ordering, initialized as *n*+1

Reverse Topological Ordering using DFS Skeleton - Recursion

```
void dfsTopo(IntList[] adjVertices, int[] color, int v, int[ ] topo, int
    topoNum)
    int w; IntList remAdj; color[v]=gray;  remAdj=adjVertices[v];
    while (remAdj≠nil)
        w=first(remAdj);
        if (color[w]==white)
            dfsTopo(adjVertices, color, w, topo, topoNum);
        remAdj=rest(remAdj);
    topoNum++; topo[v]=topoNum
    color[v]=black;
    return;
```

Obviouly, in $\Theta(m+n)$

Filling *topo* is a post-order processing, so, the earlier discovered vertex has relatively greater topo number

Correctness of the Algorithm

- If G is a DAG with *n* vertices, the procedure *dfsTopoSweep* computes a reverse topological order for G in the array *topo*.
- Proof
  - The procedure dfsTopo is called exactly once for a vertex, so, the numbers in *topo* must be distinct in the range 1,2,…*n*.
  - For any edge vw, vw can't be a back edge(otherwise, a cycle is formed). For any other edge types, we have *finishTime*(v)>*finishTime*(w), so, *topo*(w) is assigned earlier than *topo*(v). Note that *topoNum* is incremented monotonically, so, *topo*(v)>*topo*(w).

Existence of Topological Order - A Better Result

- In fact, the proof of correctness of topological ordering has proved that: DAG always has a topological order.

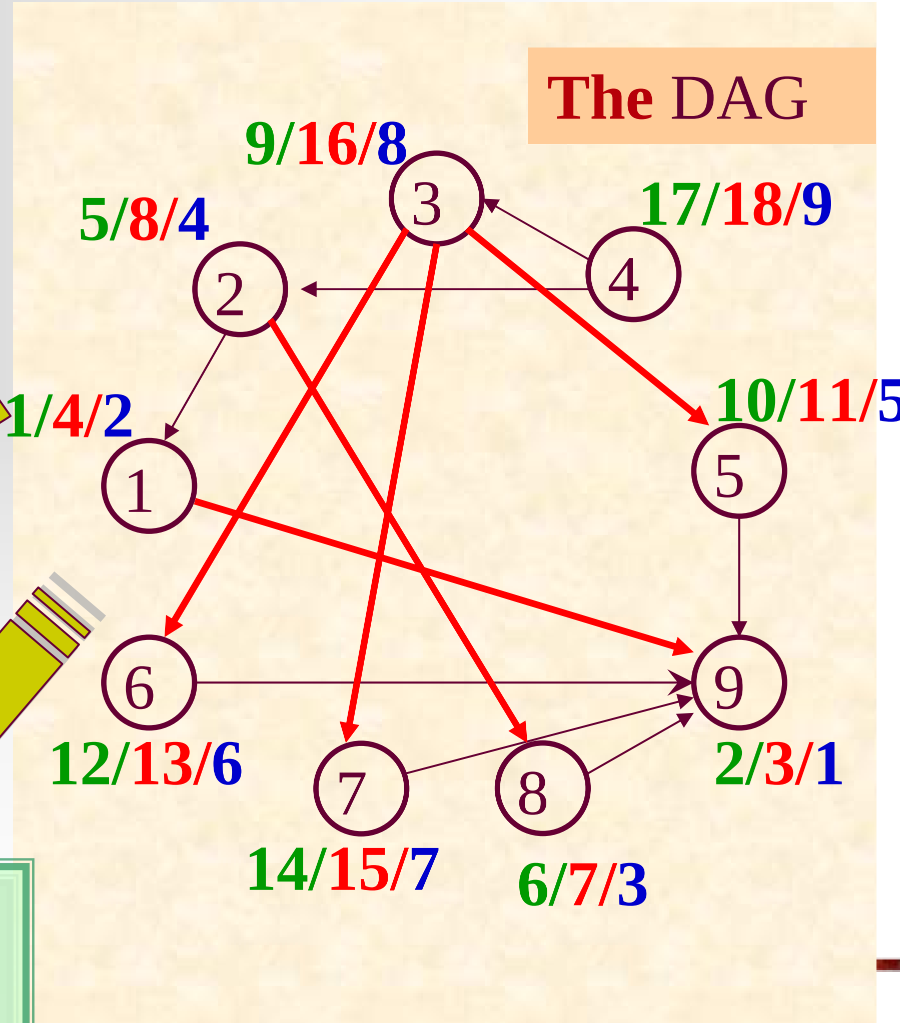
- So, **G has a topological ordering, if and only if G is a directed acyclic graph**.

Task Scheduling

- Problem: Scheduling a project consisting of a set of **interdependent** tasks to be done by one person.
- Solution:
  - Establishing a dependency graph, the vertices are tasks, and edge vw is included iff. the execution of v depends on the completion of w,
  - Making task scheduling according to the topological order of the graph(if existing).

Task Scheduling: an Example

Tasks(No.)	Depends on
choose clothes(1)	9
dress(2)	1,8
eat breakfast(3)	5,6,7
leave(4)	2,3
make coffee(5)	9
make toast(6)	9
pour juice(7)	9
shower(8)	9
wake up(9)	-



A reverse topological order

9, 1, 8, 2, 5, 6, 7, 3, 4

Critical Path in a Task Graph

- Earliest start time(est) for a task v
  - If v has no dependencies, the *est* is 0
  - If v has dependencies, the *est* is the maximum of the earliest finish time of its dependencies.
- Earliest finish time(eft) for a task v
  - For any task: *eft* = *est* + *duration*
- Critical path in a project is a sequence of tasks: $v_0, v_1, \ldots, v_k$, satisfying:
  - v_0 has no dependencies;
  - For any $v_i (i=1,2,\ldots,k)$, v_{i-1} is a dependency of v_i, such that *est* of v_i equals *eft* of v_{i-1};
  - *eft* of v_k, is maximum for all tasks in the project.

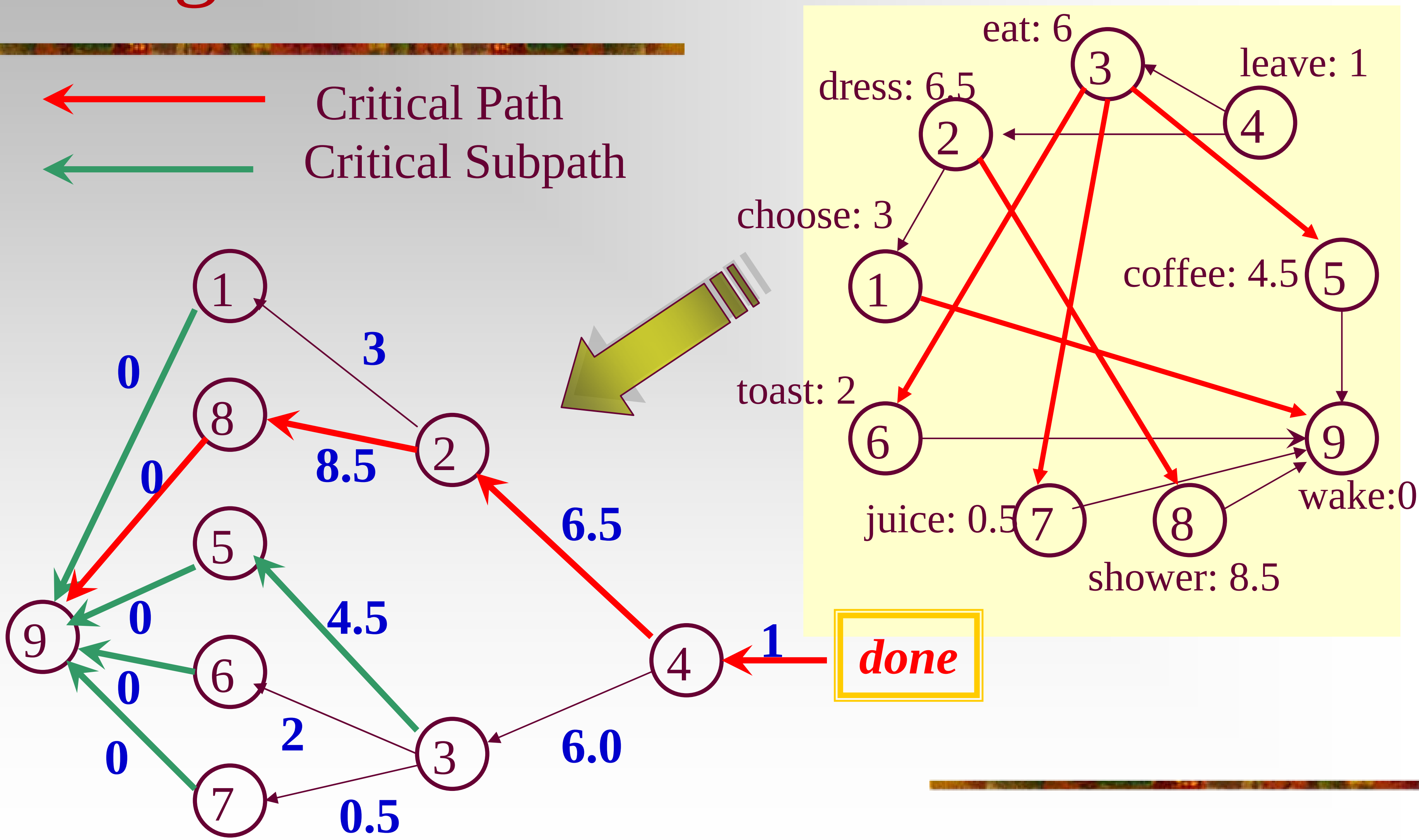
Project Optimization Problem

Assuming that parallel executions of tasks are possible except for prohibited by interdependency.

- Oberservation
  - In a critical path, v_{i-1}, is a critical dependency of v_i, i.e. any delay in v_{i-1} will result in delay in v_i.
  - The time for entire project depends on the time for the critical path.
  - Reducing the time of a off-critical-path task is no help for reducing the total time for the project.
- The problems
  - Find the critical path in a **DAG**
  - (And try to reduce the time for the critical path)

This is a precondition.

Weighted DAG with *done* Vertex



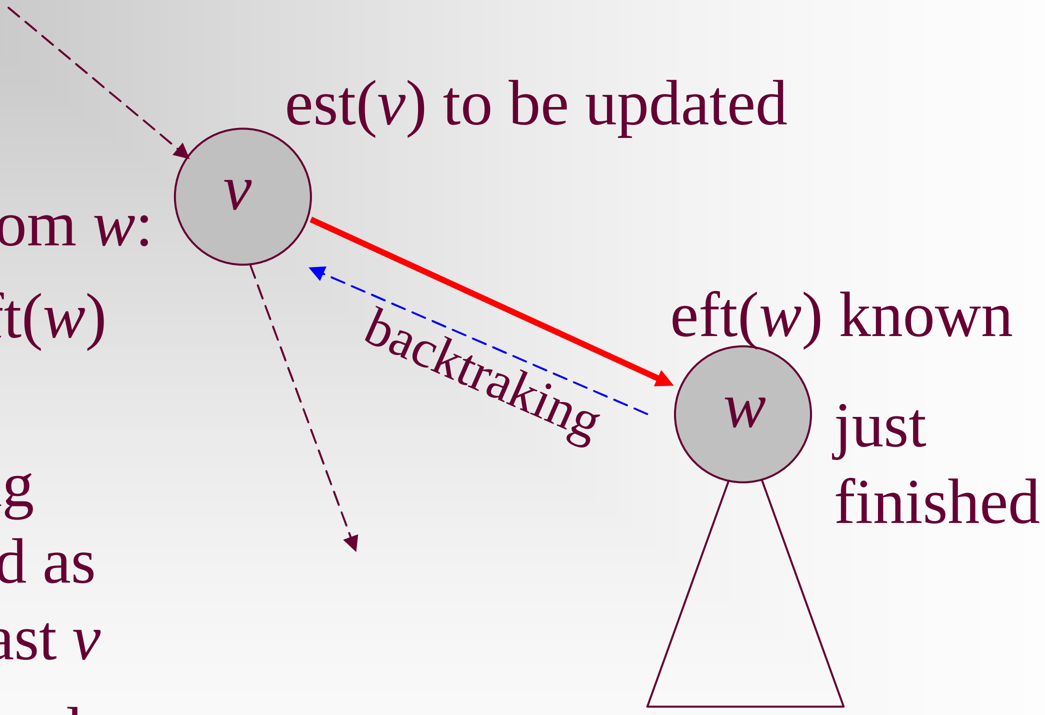
Critical Path Finding using DFS - Parameters

- Specialized parameters
  - Array *duration*, keeps the execution time of each vertex.
  - Array *critDep*, keeps the critical dependency of each vertex.
  - Array *eft*, keeps the earliest finished time of each vertex.
- Output
  - Array *topo*, *critDep*, *eft* as filled.
- Critical path is built by tracing the output.

Build the Critical Path – Case 1

Upon backtracking from *w*:

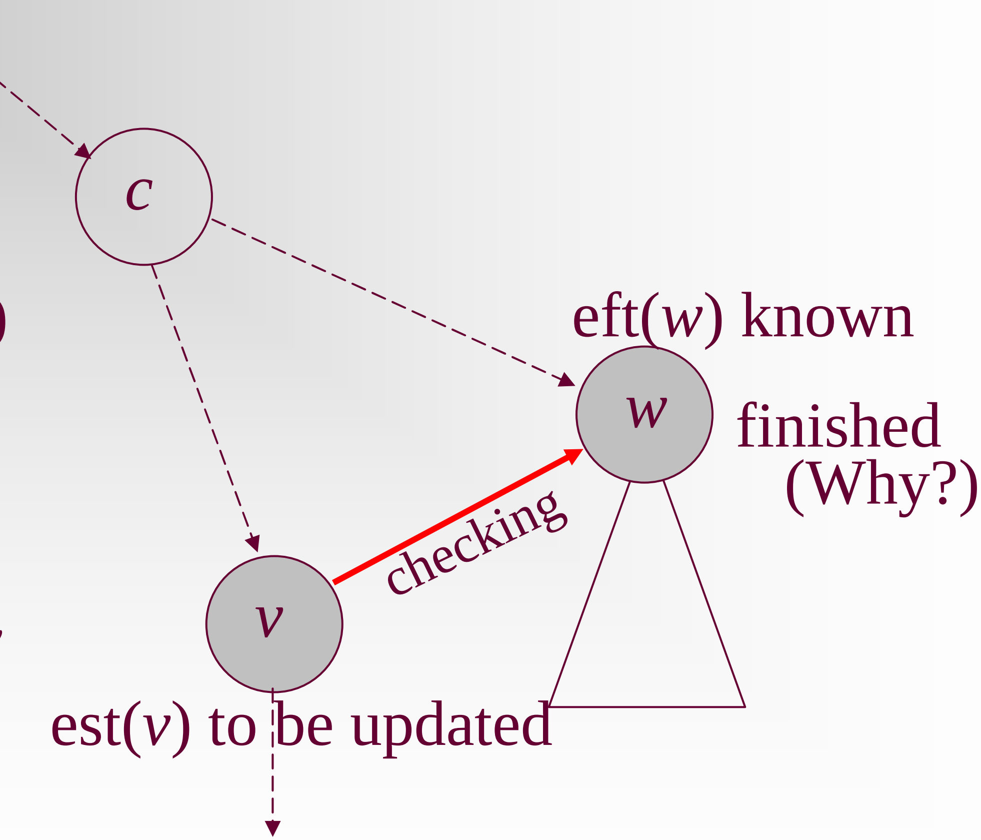
- est(*v*) is updated if eft(*w*) is larger than est(*v*)
- and the path including edge *vw* is recognized as the critical path for tast *v*
- and the eft(*v*) is updated accordingly



Build the Critical Path – Case 2

Checking *w*:

- est(*v*) is updated if eft(*w*) is larger than est(*v*)
- and the path including edge *vw* is recognized as the critical path for tast *v*
- and the eft(*v*) is updated accordingly



Critical Path Finding using DFS - Wrapper

- **void** dfsCritSweep(IntList[ ] *adjVertices*,**int** n, **int**[ ] *duration*, **int**[ ] *critDep*, **int**[ ] *eft*)
- <Allocate color array and initialize to white>
- For each vertex *v* of G, in some order
- **if** (color[v]==white)
- **dfsCrit(*adjVertices*, color, v, duration, critDep, eft);**
- // *Continue loop*
- **return**;

Critical Path Finding using DFS - Recursion

- **void** dfsCrit(.. *adjVertices*, .. color, .. v, **int**[ ] *duration*, **int**[ ] *critDep*, **int**[ ] *eft*)
- **int** w; IntList remAdj; **int est=0;**
- color[v]=gray; **critDep[v]=-1;** remAdj=*adjVertices*[v];
- **while** (remAdj≠nil) w=first(remAdj);
- **if** (color[w]==white)
- **dfsTopo(*adjVertices*, color, w, duration, critDep, efs);**
- if (eft[w]≥est) est=eft[w]; critDep[v]=w
- **else**//checking for nontree edge
- if (eft[w]≥est) est=eft[w]; critDep[v]=w
- remAdj=rest(remAdj);
- **eft[v]=est+duration[v];** color[v]=black;
- **return**;

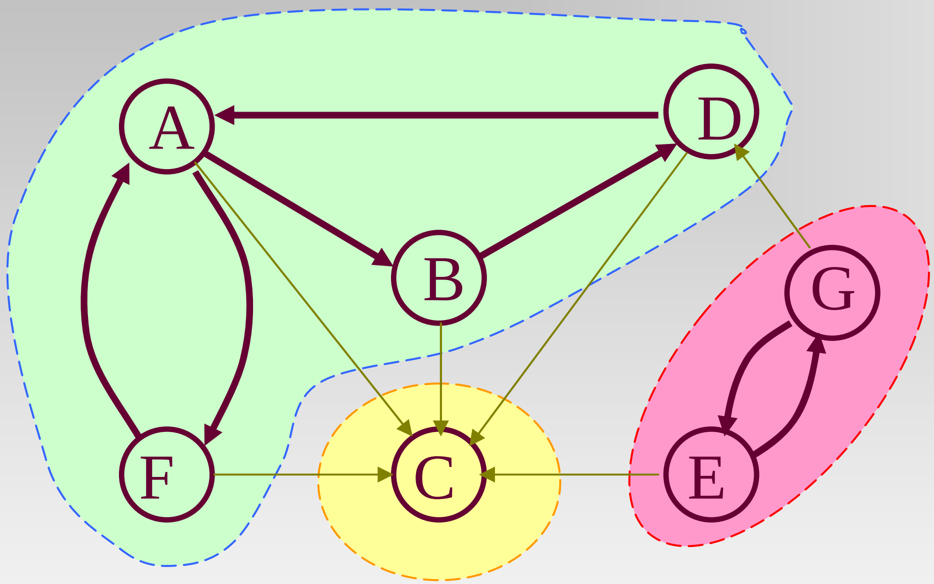
When is the eft[w] initialized?

Only black vertex

Analysis of Critical Path Algorithm

- Correctness:
  - When *eft*[*w*] is accessed in the while-loop, the w must not be gray(otherwise, there is a cycle), so, it must be black, with *eft* initialized.
  - According to DFS, each entry in the *eft* array is assigned a value exactly once. The value satisfies the definition of *eft*.
- Complexity
  - Simply same as DFS, that is $\Theta(n+m)$.

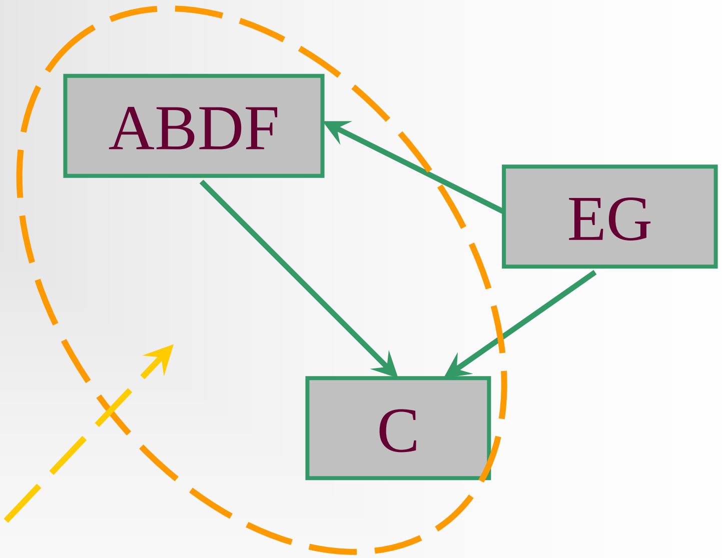
Strongly Connected and Condensation



Graph G

3 Strongly Connected Components

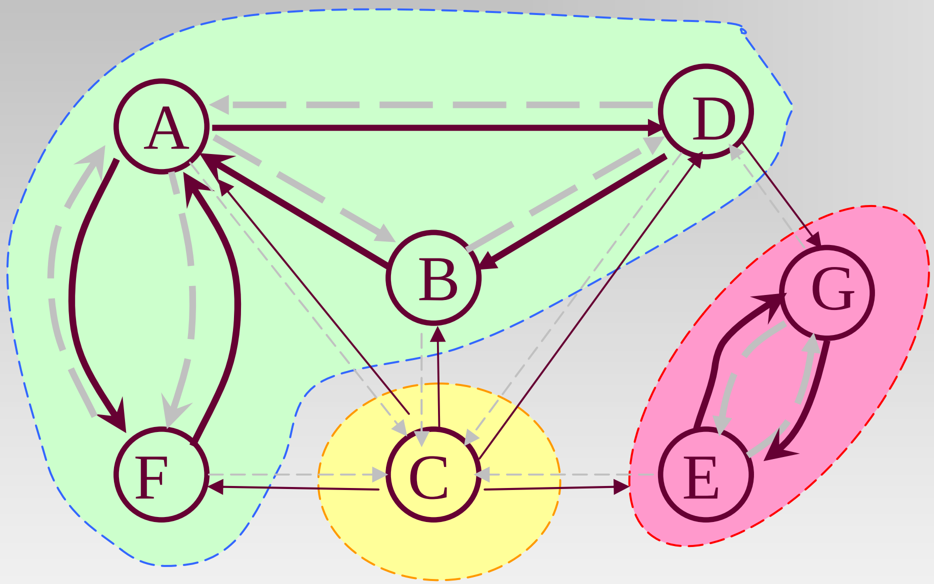
Condensation Graph G↓



It's acyclic, ***Why?***

Note: two SCC in one DFS tree

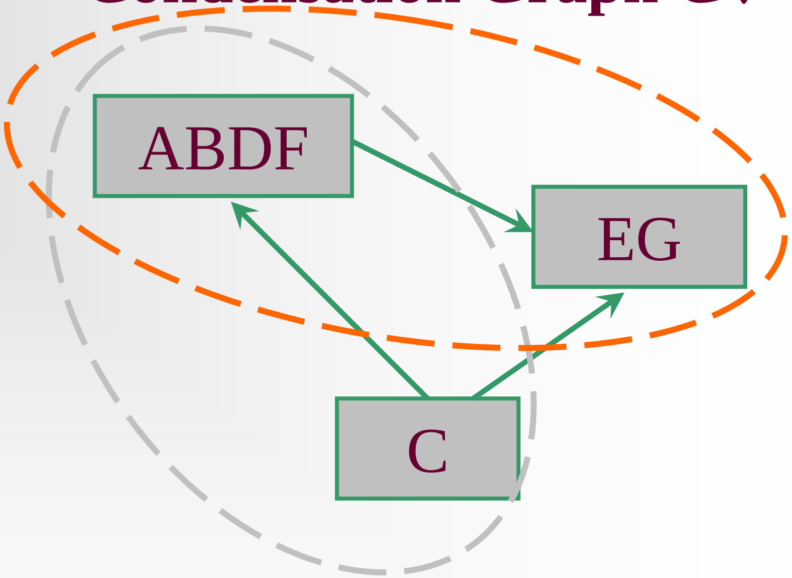
Transpose Graph



Tranpose Graph G^T

Connected Components unchanged according to vertices

Condensation Graph G↓



But, DFS tree changed

Leader of a Strong Component

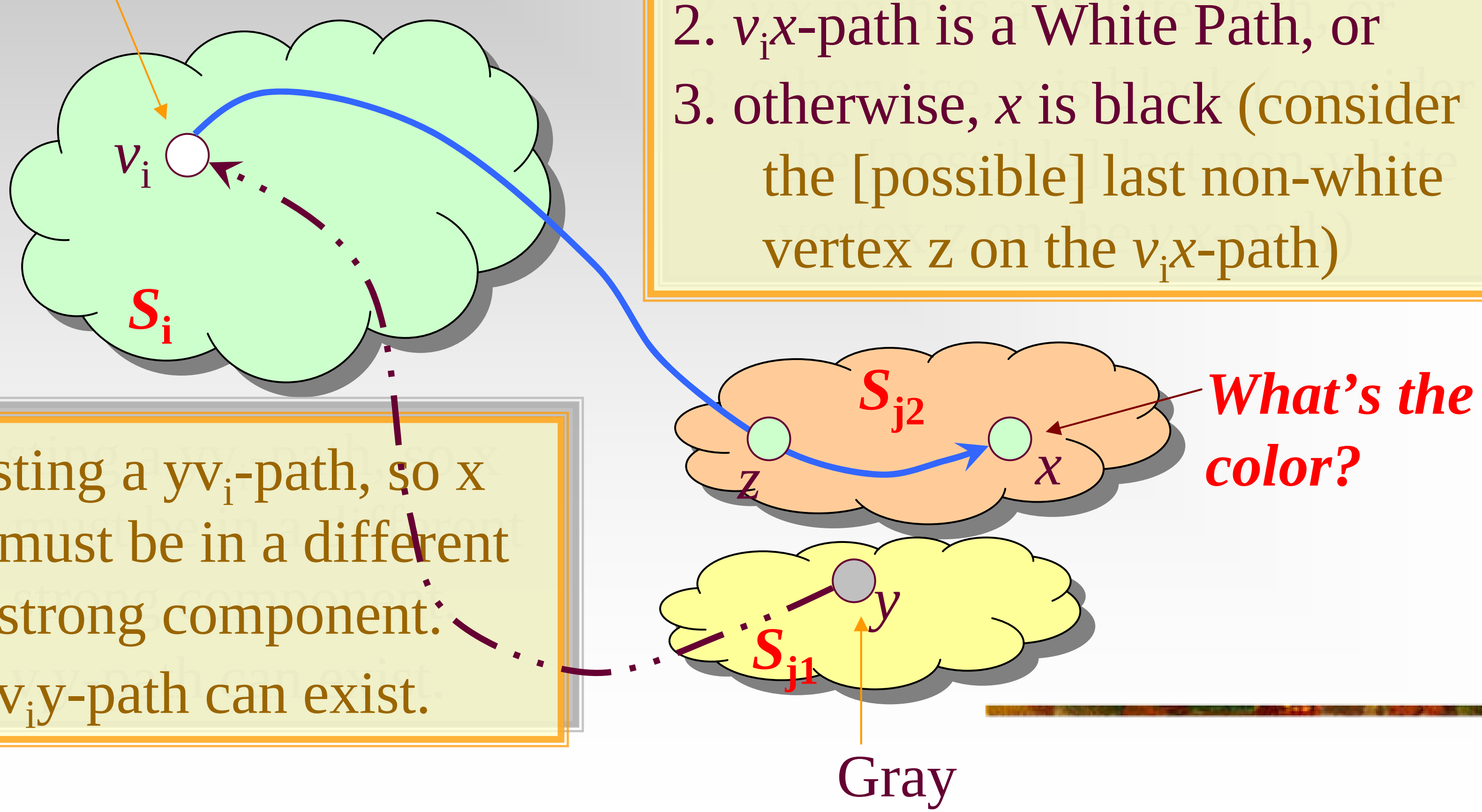
- For a DFS, the first vertex discovered in a strong component S_i is called the **leader** of S_i .
- Each DFS tree of a digraph G contains **only complete** strong components of G, one or more.
  - Proof: Applying White Path Theorem whenever the leader of S_i (i=1,2,…p) is discovered, starting with all vertices being white.
- The leader of S_i is the last vertex to finish among all vertices of S_i. (since all of them in the same DFS tree)

Path between Strong Components

The leader of S_i

At discovering

1. x can't be gray.
2. $v_i x$-path is a White Path, or
3. otherwise, x is black (consider the [possible] last non-white vertex z on the $v_i x$-path)



Existing a yv_i-path, so x must be in a different strong component.

No $v_i y$-path can exist.

Active Intervals

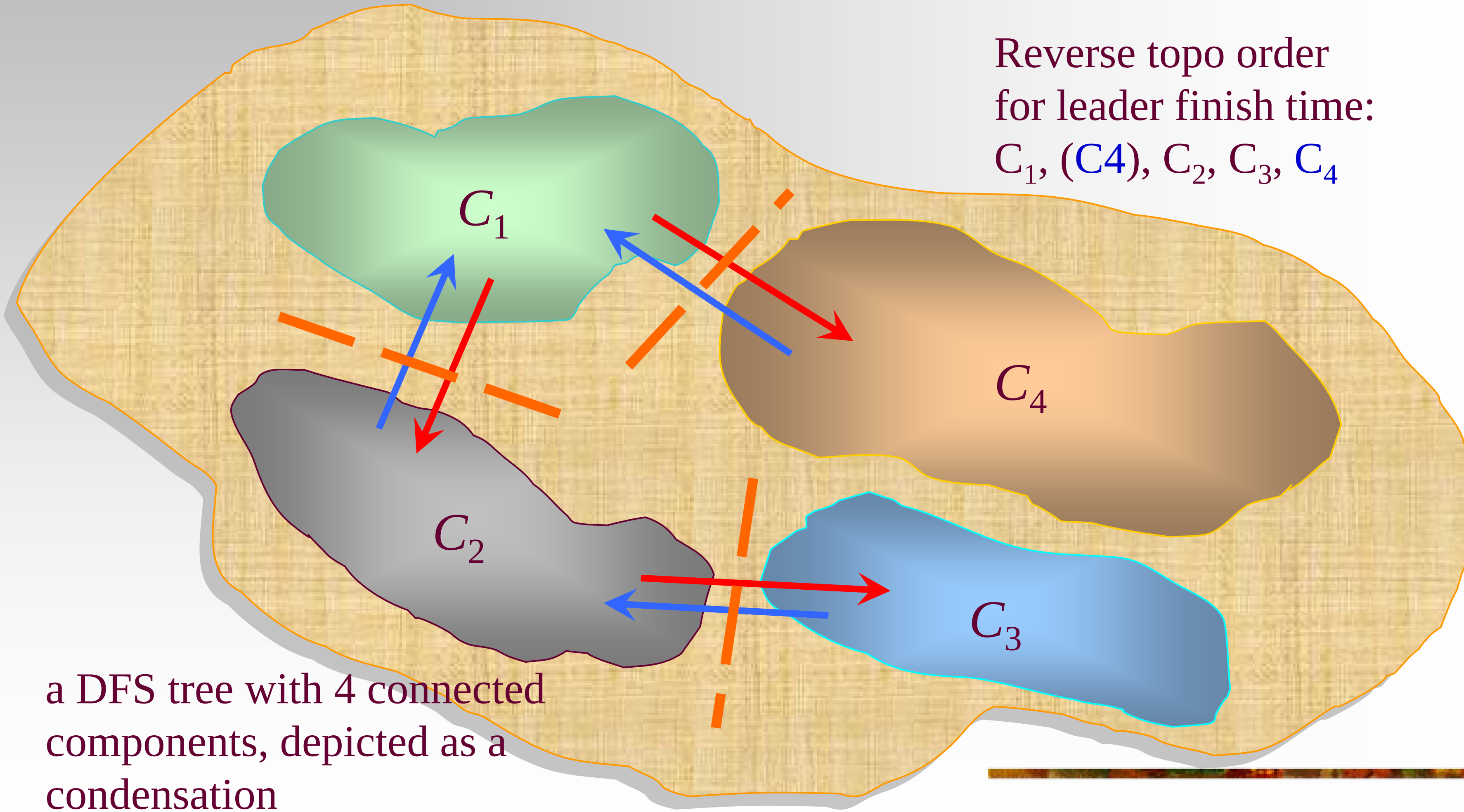
- If there is an edge from S_i to S_j, then it is **impossible** that the active interval of v_j is **entirely after** that of v_i. (Note: for leader v_i only)
  - There is no path from a leader of a strong component to any gray vertex.
  - If there is a path from the leader v of a strong component to any x in a different strong component, v finishes later than x.

Basic Idea of SCC

exploring

backtracking

Reverse topo order
for leader finish time:
C_1, (C4), C_2, C_3, C_4



a DFS tree with 4 connected components, depicted as a condensation

Strong Component Algorithm: Outline

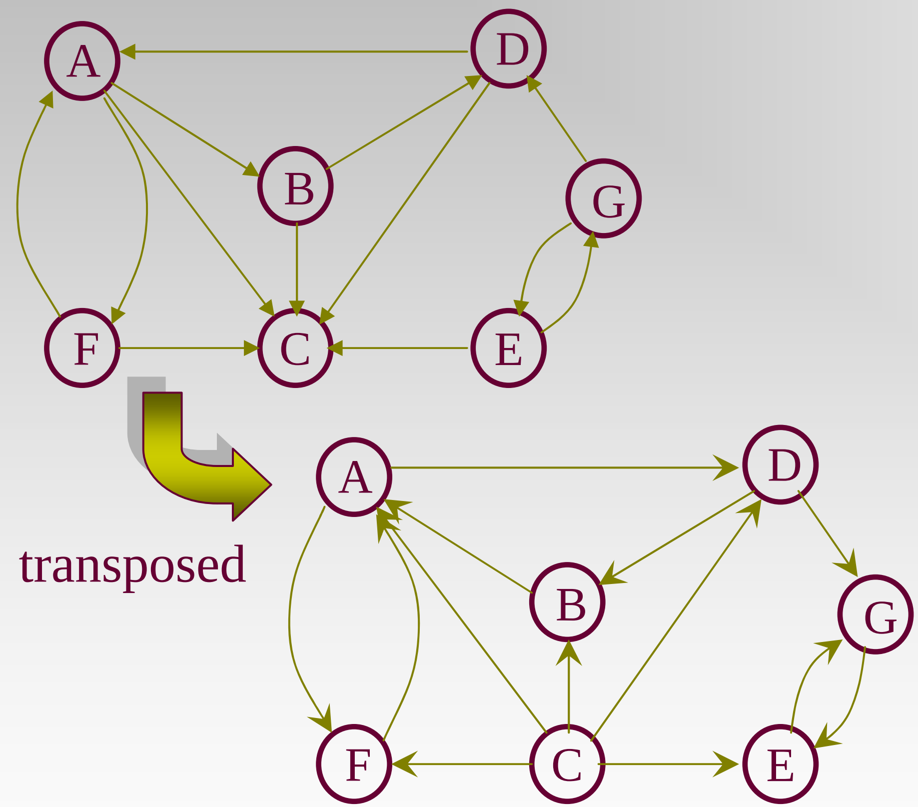
- **void** strongComponents(IntList[] *adjVertices*, **int** *n*, **int**[] *scc*)
- //*Phase* 1
- 1. IntStack *finishStack*=create(*n*);
- 2. Perform a depth-first search on *G*, using the DFS skeleton. At postorder processing for vertex *v*, insert the statement: **push(*finishStack*, *v*)**
- //*Phase* 2
- 3. Compute G^T, the transpose graph, represented as array *adjTrans* of adjacency list.
- 4. **dfsTsweep(*adjTrans*, *n*, *finishStack*, *scc*)**;
- **return**

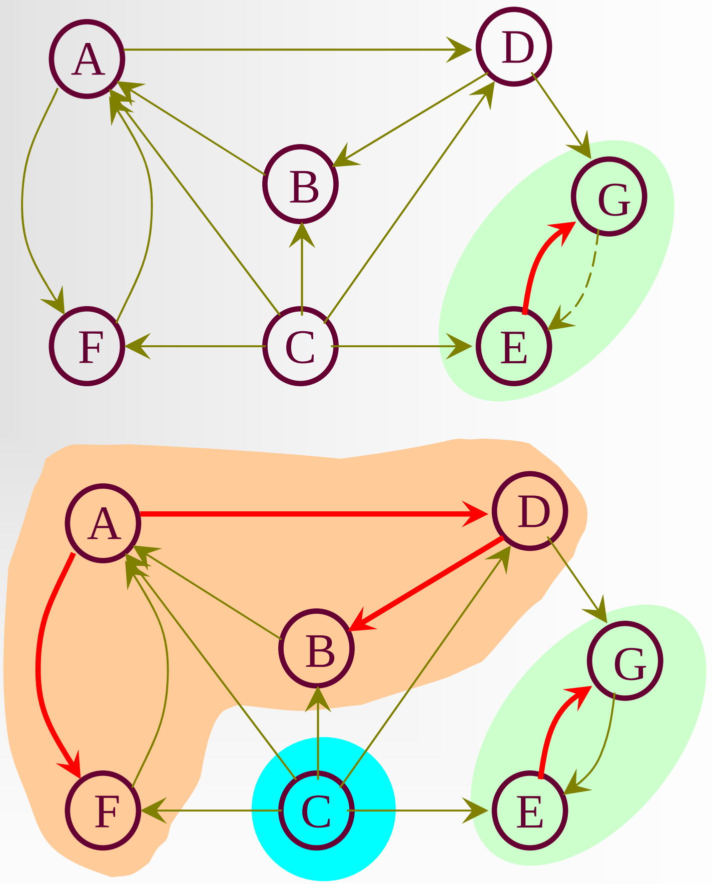
Note: *G* and G^T have the same SCC sets

Strong Component Algorithm: Core

- **void** dfsTsweep(IntList[] *adjTrans*, **int** *n*, IntStack *finishStack*, **int**[] *scc*)
- <Allocate *color* array and initialize to white>
- **while** (finishStack is not empty)
- **int** v=top(*finishStack*);
- pop(*finishStack*);
- **if** (color[*v*]==white)
- dfsT(*adjTrans*, *color*, *v*, *v*, *scc*);
- return;
- **void** dfsT(IntList[] *adjTrans*, **int**[] *color*, **int** *v*, **int** *leader*, **int**[] *scc*)
- Use the standard depth-first search skeleton. At postorder processing for vertex v insert the statement:
- scc[v]=leader;
- Pass leader and scc into recursive calls.

SCC: an Example





C D B F A G E

push/pop

Correctness of Strong Component Algorithm(1)

- In phase 2, each time a white vertex is popped from *finishStack*, that vertex is the Phase 1 leader of a strong component.
  - The later finished, the earlier popped
  - The leader is the first to get popped in the strong component it belongs to
  - If x popped is not a leader, then some other vertex in **the** strong component has been visited previously. But not a partial strong component can be in a DFS tree, so, x must be in a completed DFS tree, and is not white.

Correctness of Strong Component Algorithm(2)

- In phase 2, each depth-first search tree contains exactly one strong component of vertices
  - Only “exactly one” need to be proved
  - Assume that v_i, a phase 1 leader is popped. If another component S_j is reachable from v_i in G^T, there is a path in G from v_j to v_i. So, in phase 1, v_j finished later than v_i, and popped earlier than v_i in phase 2. So, when v_i popped, all vertices in S_j are black. So, S_j are not contained in DFS tree containing $v_i(S_i)$.

Home Assignment

- pp.378-
  - 7.17
  - 7.22
  - 7.25
  - 7.26