Reachability as Transitive Closure

Algorithm : Design & Analysis

[15]

In the last class…

- Optimization Problem
- MST Problem
  - Prim's Algorithm
  - Kruskal's Algorithm
- Single-Source Shortest Path Problem
  - Dijstra's Algorithm
- Greedy Strategy

Reachability as Transitive Closure

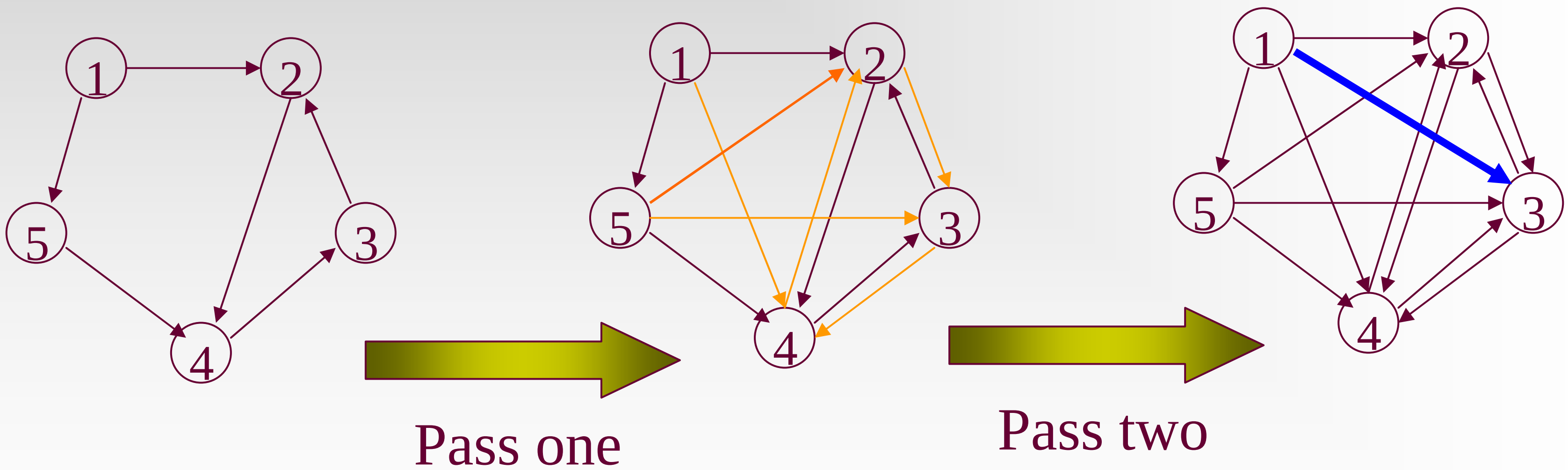
- Shortest Path and Transitive Closure
- Washall's Algorithm for Transitive Closure
- All-Pair Shortest Paths
- Matrix for Transitive Closure
- Multiplying Bit Matrices - Kronrod's Algorithm

Fundamental Questions

- For **all** pair of vertices in a graph, say, u, v:
  - Is there a path from u to v?
  - What is the shortest path from u to v?
- Reachability as a (reflexive) transitive closure of the adjacency relation, which can be represented as a bit matrix.

Transitive Closure by Shortcuts

- The idea: if there are edges $s_i s_k$, $s_k s_j$, then an edge $s_i s_j$, the “shortcut” is inserted.



Note: the triple (1,5,3) is considered more than once

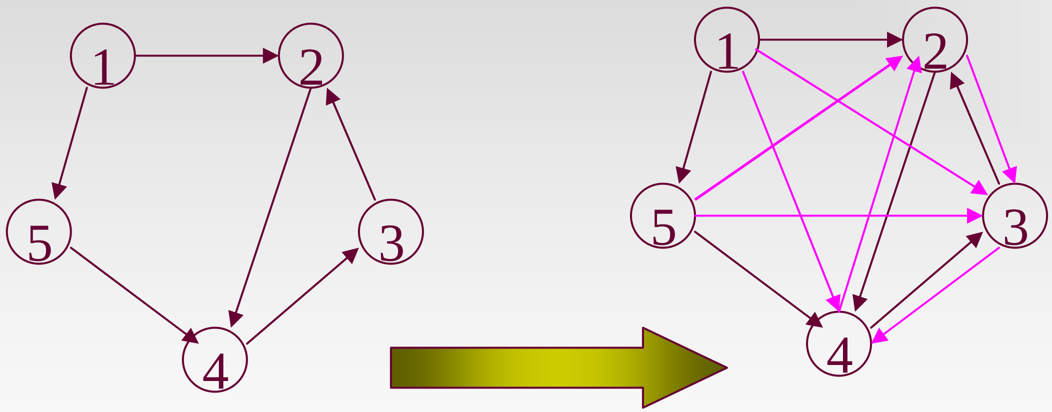
Trans. Closure by Shortcuts: algorithm

- Input: A, an $n \times n$ **boolean** matrix that represents a binary relation
- Output: R, the **boolean** matrix for the transitive closure of A
- Procedure
  - **void** simpleTransitiveClosure(**boolean**[ ][ ] A, **int** n, **boolean**[ ][ ] R)
  - **int** i,j,k;
  - Copy A to R;
  - Set all main diagonal entries, r_{ii}, to *true*;
  - **while** (any entry of R changed during one complete pass)
  - **for** (i=1; i≤n; i++)
  - **for** (j=1; j≤n; j++)
  - **for** (k=1; k≤n; k++)
  - $r_{ij}=r_{ij}\vee(r_{ik}\wedge r_{kj})$

The order of (i,j,k) matters

Shortcuts in different order

- Duplicated checking may be deleted by changing the order of the vertices.



No edge is added in Pass two. End.

Pass one

Check the vertices in decreasing order.

Change the order: Washall's Algorithm

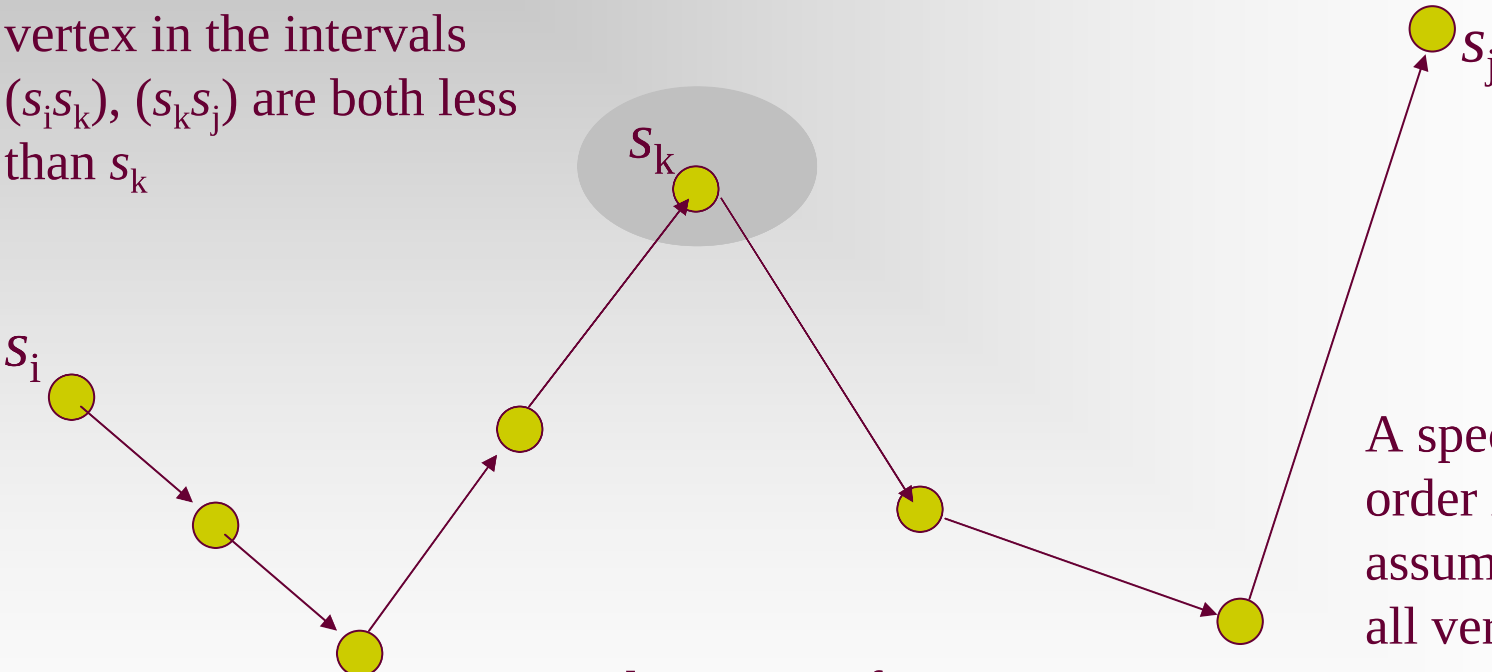
- **void** simpleTransitiveClosure(**boolean**[][] A, **int** *n*, **boolean**[][] *R*)
- **int** i,j,k;
- Copy *A* to *R*;
- Set all main diagonal entries, r_{ii}, to *true*;
- ~~**while** (any entry of *R* changed during one complete pass)~~
- **for** (k=1; k≤*n*; k++)
- **for** (i=1; i≤*n*; i++)
- **for** (j=1; j≤*n*; j++)
- $r_{ij}=r_{ij}\vee(r_{ik}\wedge r_{kj})$

k varys in the outmost loop

Note: "false to true" can not be reversed

Highest-numbered intermediate vertex

The highest intermediate vertex in the intervals $(s_i s_k)$, $(s_k s_j)$ are both less than s_k



A specific order is assumed for all vertices

Vertical position of vertices reflect their vertex numbers

Correctness of Washall's Algorithm

- Notation:
  - The value of r_{ij} changes during the execution of the body of the "**for** *k*…" loop
    - After initializations: $\boldsymbol{r_{ij}^{(0)}}$
    - After the *k*th time of execution: $\boldsymbol{r_{ij}^{(k)}}$

Correctness of Washall's Algorithm

- If there is a simple path from s_i to s_j($i \neq j$) for which the highest-numbered intermediate vertex is s_k, then $r_{ij}^{(k)}$=true.
- Proof by induction:
  - Base case: $r_{ij}^{(0)}$=true if and only if $s_i s_j \in E$
  - Hypothesis: the conclusion holds for $h<k$($h \geq 0$)
  - Induction: the simple $s_i s_j$-path can be looked as $s_i s_k$-path+$s_k s_j$-path, with the indices h_1, h_2 of the highest-numbered intermediate vertices of both segment **strictly**(simple path) less than k. So, $r_{ik}^{(h1)}$=true, $r_{kj}^{(h2)}$=true, then $r_{ik}^{(k-1)}$=true, $r_{kj}^{(k-1)}$=true(Remember, false to true can not be reversed). So, $r_{ij}^{(k)}$=true

Correctness of Washall's Algorithm

- If there is ***no*** path from s_i to s_j, then r_{ij}=*false*.
- Proof
  - If r_{ij}=*true*, then only two cases:
  - r_{ij} is set by initialization, then $s_i s_j \in E$
  - Otherwise, r_{ij} is set during the *k*th execution of (**for** *k*=1,2,…) when $r_{ik}^{(k-1)}$=true, $r_{kj}^{(k-1)}$=true, which, recursively, leading to the conclusion of the existence of a $s_i s_j$-path. (Note: If a $s_i s_j$-path exists, there exists a simple $s_i s_j$-path)

All-pairs Shortest Path

- Non-negative weighted graph
- Shortest path property: If a shortest path from x to z consisting of path P from x to y followed by path Q from y to z. Then P is a shortest xz-path, and Q, a shortest zy-path.
- The regular matrix representing a graph can easily be transformed into a (minimum) distance matrix D

(just replacing 1 by edge weight, 0 by infinity,

and setting main diagonal elements as 0)

Computing the Distance Matrix

- Basic formula:
  - $D^{(0)}[i][j]=w_{ij}$
  - $D^{(k)}[i][j]=\min(D^{(k-1)}[i][j], D^{(k-1)}[i][k]+ D^{(k-1)}[i][j])$

- Basic property:
  - $D^{(k)}[i][j]\le d_{ij}^{(k)}$

    where $d_{ij}^{(k)}$ is the weight of a shortest path from v_i to v_j with highest numbered intermediate vertex v_k.

All-Pairs Shortest Paths

- Floyd algorithm
  - Only slight changes on Washall's algorithm.

```
Void allPairsShortestPaths(float [][] W, int n, float [][] D)
    int i, j, k;
    Copy W into D;
    for (k=1; k≤n; k++)
        for (i=1; i≤n; i++)
            for (j=1; j≤n; j++)
                D[i][j] = min (D[i][j], D[i][k]+D[k][j];
```

- Routing table tracking the path

Matrix Representation

- Define family of matrix $A^{(p)}$:
  - $a_{ij}^{(p)}=true$ if and only if there is a path of length p from s_i to s_j.
- $A^{(0)}$ is specified as identity matrix. $A^{(1)}$ is exactly the adjacency matrix.
- Note that $a_{ij}^{(2)}=true$ if and only if exists some s_k, such that both $a_{ik}^{(1)}$ and $a_{kj}^{(1)}$ are *true*. So, $a_{ij}^{(2)}=\bigvee_{k=1,2,\ldots,n}(a_{ik}^{(1)}\wedge a_{kj}^{(1)})$, which is an entry in the *Boolean matrix product*.

Boolean Matrix Operations: Recalled

- Boolean matrix prodect $C=AB$ as:
  - $c_{ij}=\bigvee_{k=1,2,\ldots,n}(a_{ik}\wedge b_{kj})$
- Boolean matrix sum $D=A+B$ as:
  - $d_{ij}=a_{ik}\vee b_{kj}$
- R, the transitive closure matrix of A, is the sum of all A^p, p is a non-negative integer.
- For a digraph with n vertices, the length of the longest simple path is no larger than n-1.

Bit Matrix

- A **bit string** of length n is a sequence of n bits occupying contiguous storage(word boundary) (usually, n is larger than the word length of a computer)
- If A is a **bit matrix** of $n \times n$, then $A[i]$ denotes the ith row of A which is a bit string of length n. a_{ij} is the jth bit of $A[i]$.
- The **procedure** bitwiseOR(a,b,n) compute $a \vee b$ bitwise for n bits, leaving the result in a.

Straightforward Multiplication of Bit Matrix

- Computing C=AB
  - <Initialize C to the zero matrix>
  - **for** (i=1; i≤n, i++)
  -     **for** (k=1; k≤n, k++)
  -         **if** (a_{ik}==*true*) bitwiseOR($C[i]$, $B[k]$, n)

Thought as a union of sets (row union), n^2 unions are done at most

In the case of a_{ik} is *true*, c_{ij}=$a_{ik}b_{kj}$ is true iff. b_{kj} is true. As a result: $C[i]=\cup_{k\in A[i]}B[k]$, $(A[i]=\{k|ik=true\})$

Union for B[k] is **repeated each time** when the kth bit is *true* in a different row of A is encountered.

Reducing the Duplicates by Grouping

- Multiplication of A, B, two 12×12 matrices

$$
\begin{array}{c}
A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12}
\end{array}
\left(
\begin{array}{cccc|cccccccc}
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & & & & \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & 0 & 1 & 0 & 1 & & & & \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & & & & &
\end{array}
\right)
$$

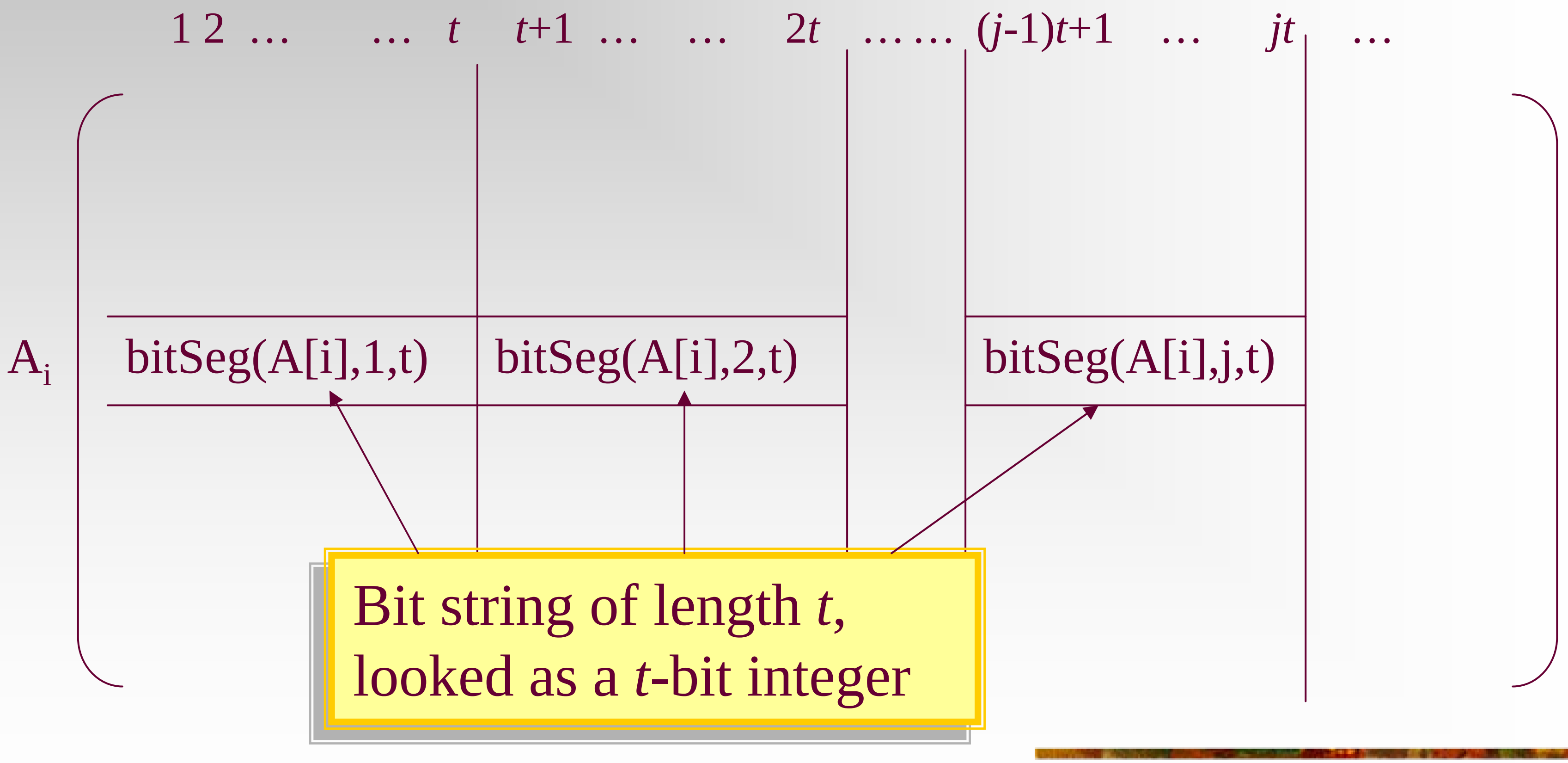
Segment

Length t

- 12 rows of B are divided evenly into 3 groups, with rows 1-4 in group 1, etc.
- With each group, all possible unions of different rows are pre-computed. (This can be done with 11 unions if suitable order is assumed.)
- When the first row of AB is computed, ($B[1] \cup B[3] \cup B[4]$) is used in stead of 3 different unions, and this combination is used in computing the 3^{rd} and 7^{th} rows as well.

The Segmentation for Matrix A

The $n \times n$ array



An Example

bitSeg(A[7], 1, t)

$= 1011_2 = 11$

Group 1 $(1\ldots t)$	Group 2 $(t+1\ldots 2t)$	Group 3 $(2t+1\ldots 3t)$

$$
\begin{array}{c}
A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12}
\end{array}
\begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
$$

$t=4$

B_1, B_2, B_3, B_4	any union combination of $\{B_1,B_2,B_3,B_4\}$, totaling 16, 11 computed	Group 1 $(1\ldots t)$
B_5, B_6, B_7, B_8	any union combination of $\{B_5,B_6,B_7,B_8\}$, totaling 16, 11 computed	Group 2 $(t+1\ldots 2t)$
B_9, B_{10}, B_{11}, B_{12}	any union combination of $\{B_9,B_{10},B_{11},B_{12}\}$, totaling 16, 11 computed	Group 3 $(2t+1\ldots 3t)$

Where to store?

Storage of the Row Combinations

- Using one large 2-dimensional array
- Goals
  - keep all unions generated
  - provide indexing for using
- Coding within a group
  - One-to-one correspondence between a bit string of length t and one union for a subset of a set of t elements
- Establishing indexing for union required
  - When constructing a row of AB, a segment can be notated as a integer. Use it as index.

Storage the Unions

allUnion

one row for
one group

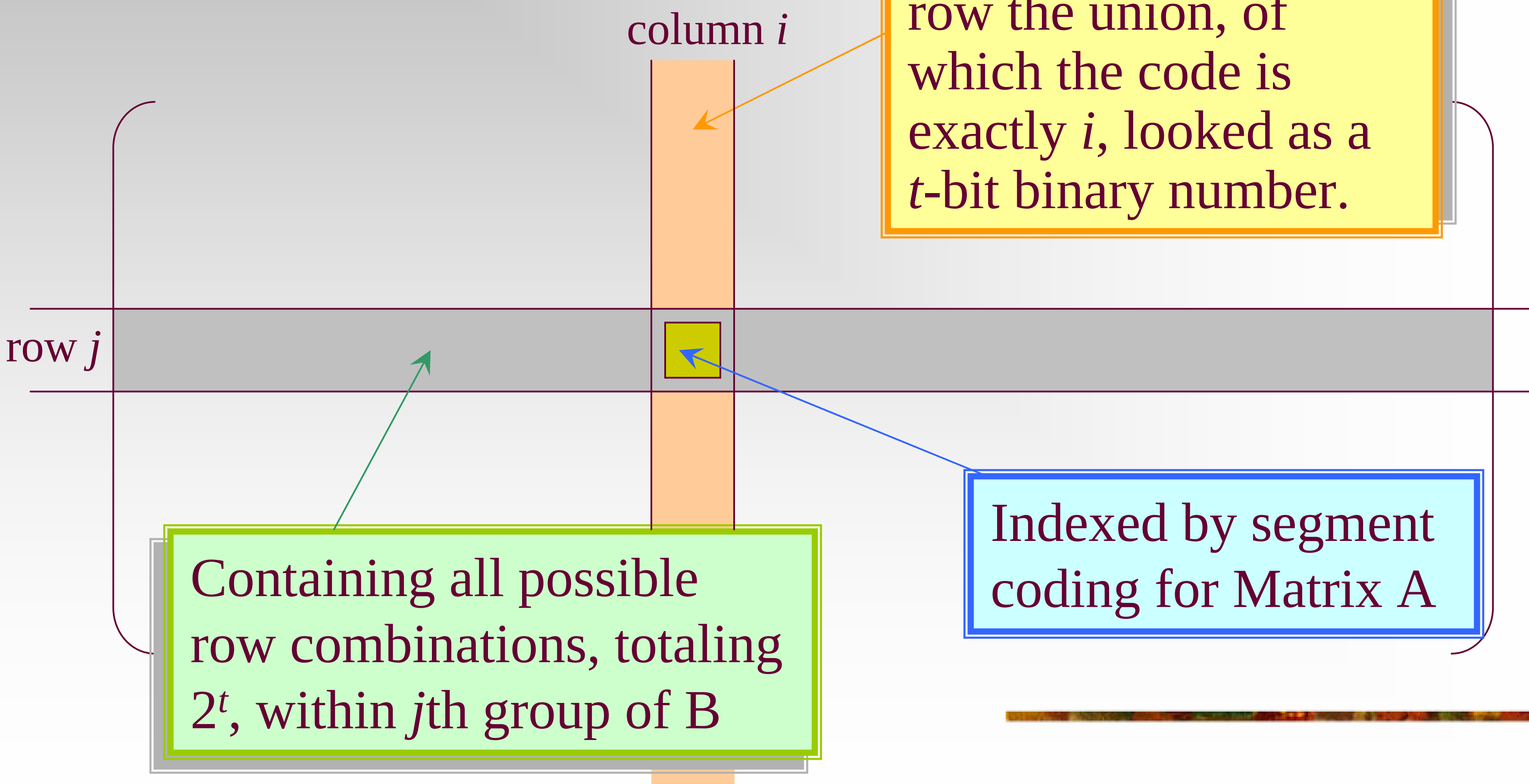
column indexed by bitSeg(A[i],j,t)

$$\begin{bmatrix} \phi & 4 & 3 & 3{,}4 & 2 & 2{,}4 & 2{,}3 & 2{,}3{,}4 & 1 & 1{,}4 & 1{,}3 & 1{,}3{,}4 & 1{,}2 & 1{,}2{,}4 & 1{,}2{,}3 & 1{,}2{,}3{,}4 \\ \phi & 8 & 7 & 7{,}8 & 6 & 6{,}8 & & & & & \cdots & \cdots & & & & \\ \phi & 12 & 11 & 11{,}12 & 10 & 10{,}12 & & & & & & & & & & \end{bmatrix}$$

i,j,k stands for $B_i \cup B_j \cup B_k$

Array for Row Combinations

The array: **allUnions**



Cost as Function of Group Size

- Cost for the pre-computation
  - There are 2^t different combination of rows in one group, including an empty and t singleton. Note, in a suitable order, each combination can be made using only one union. So, the total number of union is **$g[2^t-(t+1)]$**, where g=*n*/t is the number of group.
- Cost for the generation of the product
  - In computing one of *n* rows of *AB*, at most one combination from each group is used. So, the total number of union is ***ng***

Selecting Best Group Size

- The total number of union done is:

  $$g[2^t-(t+1)]+n(g-1) \approx (n2^t)/t+n^2/t \ \text{(Note: } g=n/t\text{)}$$

- Trying to minimize the number of union
  - Assuming that the first term is of higher order:
    - Then $t \geq \lg n$, and the least value is reached when $t=\lg n$.
  - Assuming that the second term is of higher order:
    - Then $t \leq \lg n$, and the least value is reached when $t=\lg n$.
- So, when $t \approx \lg n$, the number of union is roughly $2n^2/\lg n$, which is of lower order than n^2. We use $\mathbf{t=\lfloor \lg n \rfloor}$

For symplicity, exact power for n is assumed

Scketch for the Procedure

- t=$\lfloor \lg n \rfloor$; g=$\lceil n/t \rceil$;
- <Compute and store in allUnions unions of all combinations of rows of *B*>
- **for** (i=1; i≤*n*; i++)
-     <Initialize C[i] to 0>
-     **for** (j=1; j≤*g*; j++)
-         C[i] = C[i] ∪ allUnions[j][bitSeg(A[i],j,t)]

Kronrod Algorithm

- Input: A,B and n, where A and B are n×n bit matrices.
- Output: C, the Boolean matrix product.
- Procedure
  - The processing order has been changed, from “row by row” to “group by group”, resulting the reduction of storage space for unions.

Complexity of Kronrod Algorithm

- For computing all unions within a group, 2^t-1 union operations are done.
- One union is bitwiseOR'ed to n row of C
- So, altogether, $(n/t)(2^t-1+n)$ row unions are done.
- The cost of row union is $\lceil n/w \rceil$ bitwise or operations, where w is word size of bitwise or instruction dependent constant.

Home Assignments

- pp.446-
  - 9.10
  - 9.12
  - 9.16
  - 9.17