NP-Complete Problems

Algorithm : Design & Analysis

[20]

In the last class…

- Simple String Matching
- KMP Flowchart Construction
- Jump at Fail
- KMP Scan

NP-Complete Problems

- Decision Problem
- The Class *P*
- The Class *NP*
- *NP*-Complete Problems
  - Polynomial Reductions
  - *NP*-hard and *NP*-complete

How Functions Grow

Algorithm	1	2	3	4	
Time function(ms)	$33n$	$46n \lg n$	$13n^2$	$3.4n^3$	2^n
Input size(n)	Solution time				
10	0.00033 sec.	0.0015 sec.	0.0013 sec.	0.0034 sec.	0.001 sec.
100	0.0033 sec.	0.03 sec.	0.13 sec.	3.4 sec.	4×10^{16} yr.
1,000	0.033 sec.	0.45 sec.	13 sec.	0.94 hr.	
10,000	0.33 sec.	6.1 sec.	22 min.	39 days	
100,000	3.3 sec.	1.3 min.	1.5 days	108 yr.	
Time allowed	Maximum solvable input size (approx.)				
1 second	30,000	2,000	280	67	20
1 minute	1,800,000	82,000	2,200	260	26

Hanoi Tower Revisited

- It is **easy** to provide a recursive algorithm to resolve the problem of Hanoi Tower. The solution requires 2^N-1 moves of disc.
- It is **extremely difficult** to achieve the result for an input of moderate size. For the input of 64, it takes half a million years even if the Tibetan priest has superhuman strength to move a million discs in a second.

Max Clique: an Example

- A maximal complete subgraph of a graph G is called a **clique**, whose size is the number of vertices in it.
  - **Optimization problem**: Find the maximal clique in a given graph G.
  - **Decision problem**: Has G a clique of size at least k for some given k?

Decision Problem

- Statement of a decision problem
  - Part 1: instance description defining the input
  - Part 2: question stating the actual yes-or-no question
- A decision problem is a mapping from all possible inputs into the set {*yes*, *no*}

Optimization vs. Decision

- Usually, a optimization problem can be rephrased as a decision problem.
- For some cases, it can be proved that the decision problem can be solved in polynomial time if and only if the corresponding optimization problem can.
- We can make the statement that if the decision problem cannot be solved in polynomial time then the corresponding optimization problem cannot either.

Max Clique Revisited

- The max clique problem can be solved in polynomial time iff. the corresponding decision problem can be solved in polynomial time.
  - If the size of a max clique can be found in time *g*(n), the corresponding decision may be settled in that time of course.
  - If deciClique is algorithm for the decision problem with *k* in the complexity of *f*(*n*), then we apply the algorithm at most *n* time, for *k*=*n*, *n*-1, ..., 2, 1, and we can solve the optimization problem, and with the complexity no worse than *nf*(*n*), which is polynomial only if *f*(*n*) is polynomial.

Some Typical Decision Problems

- Graph coloring
  - Given a undirected graph G and a positive integer k, is there a coloring of G using at most k colors?
- Job scheduling with penalties
  - Given a group of jobs, each with its execution duration, deadline and penalty for missing the deadline, and a nonnegative integer k, is there a schedule with the total penalty bounded by k?

Some Typical Decision Problems

- Bin packing
  - Given k bins each of capacity one, and n objects with size $s_1, \ldots, s_n$, (where s_i is a rational number in (0,1]). Do the n objects fit in k bins?
- Knapsack
  - Given a knapsack of capacity C, n objects with sizes $s_1, \ldots, s_n$ and "profits" $p_1, \ldots, p_n$, and a positive integer k. Is there a subset of the n objects that fits in the knapsack and has total profit at least k?

  (Subset sum as a simplified version)

Some Typical Decision Problems

- *CNF*-Satisfiability
  - Given a CNF formula, is there a truth assignment that satisfies it?
- Hamiltonian cycles or Hamiltonian paths
  - Given a undirected graph *G*. Does *G* have a Hamiltionian cycle of Hamiltonian path?
- Traveling salesperson
  - Given a complete, weighted graph and an integer *k*, is there a Hamiltonian cycle with total weight at most *k*?

Theory of *NP*-Completeness

- What it **cannot** do
  - Provide a method of obtaining polynomial time algorithms for those “hard” problems
  - Negate the existence of algorithms of polynomial complexity for those problems
- What it **can** do
  - Show that many of the problems for which there is no known polynomial time algorithm are computationally related.

The Class *P*

- A **polynomially bounded algorithm** is one with its worst-case complexity bounded by a polynomial function of the input size.
- A **polynomially bounded problem** is one for which there is a polynomially bounded algorithm.

- **The class *P* is the class of decision problems that are polynomially bounded**.

Notes on the Class *P*

- Class *P* has a too broad coverage, in the sense that not every problems in *P* has an acceptable efficient algorithm. However, the problem not in *P* must be extremely expensive and probably impossible to solve in practice.
- The problems in *P* have nice “closure” properties for algorithm integration.
- The property of being in *P* is independent of the particular formal model of computation used.

Nondeterministic Algorithm

Phase 1 Guessing: generating **arbitrarily** "certificate", i.e. proposed solution

```
void nondetA(String input)
    String s=genCertif();
    Boolean CheckOK=verifyA(input,s);
    if (checkOK)
        Output "yes";
    return;
```

The algorithm may behave differently on the same input in different executions: "yes" or "no output".

Phase 2 Verifying: determining if *s* is a valid description of a object for answer, and satisfying the criteria for solution

Answer of Nondeterministic Algorithm

- For a particular decision problem with input *x*:
  - The answer computed by a nondeterministic algorithm is defined to be *yes* if and only if there is some execution of the algorithm that gives a *yes* output.
  - The answer is *no*, if for **all** *s*, there is no output.

Nondeterministic vs. Deterministic

```
void nondetSearch(int k; int[ ] S)
    int i =genCertif();
    if (S[i]=k)
        Output "yes";
    return;
```

In ***O*(1)**

Note: $\Omega(n)$ for deterministic algorithm

```
void nondetSort(int[ ] S; int n)
    int i, j; int[ ] out=0;
    for i =1 to n do
        j= genCertif();
        if out[j ]≠0 then return;
        out[j]=S[i ];
    for i =1 to n-1 do
        if out[i ]>out[i+1] then return;
    S=out;
    Output(yes);
return
```

In ***O*(*n*)**

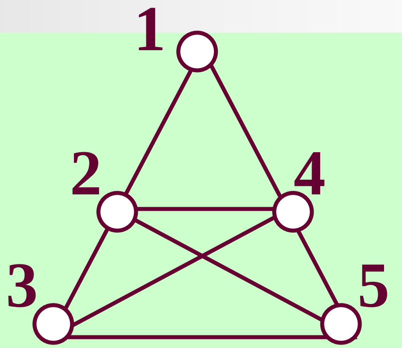
Note: $\Omega(n\log n)$ for deterministic algorithm

Nondeterministic Graph Coloring

Problem instance *G*

Input string:

4,5,(1,2)(1,4)(2,4)(2,3)(3,5)(2,5)(3,4)(4,5)



s	Output	Reason
RGRBG	*false*	v_2 and v_5 conflict
RGRB	*false*	Not all vertices are colored
RBYGO	*false*	Too many colors used
RGRBY	*true*	**A valid 4-coloring**
R%*,G@	*false*	Bad syntax

verified by phase 2

(G,4)→yes

generated by phase 1

The Class *NP*

- A **polynomial bounded nondeterministic algorithm** is one for which there is a (fixed) polynomial function p such that for each input of size n for which the answer is *yes* , there is some execution of the algorithm that produces a *yes* output in at most $p(n)$ steps.
- The **class *NP*** is the class of decision problems for which there is a polynomial bounded nondeterministic algorithm.

Deterministic Interpretation

- Allowing unbounded parallelism in computation
  - One copy of the algorithm is made for each of the possible guess
  - All the copies are executing at the same time
  - The first copy output a “yes” terminates all other computations.

Proof of Being in *NP*

- Graph coloring is in *NP*
  - Description of the input and the certificate
  - Properties to be checked for a answer "*yes*"
    - There are *n* colors listed: c_1, c_2, …, c_n (not necessarily different)
    - Each c_i is in the range 1,…,*k*
    - Scan the list of edges to see if a conflict exists
  - Proving that each of the above statement can be checked in polynomial time.

Max Clique Problem is in $\mathcal{NP}$

```
void nondeteClique(graph G; int n, k)
    set S=ϕ;
    for int i=1 to k do
        int t=genCertif();
        if t∈S then return;
        S=S∪{t };
    for all pairs (i,j) with i,j in S and i≠j do
        if (i,j ) is not an edge of G
            then return;
        Output("yes");
```

In $O(n)$

In $O(k^2)$

So, we have an algorithm for the maximal clique problem with the complexity of $O(n+k^2)=O(n^2)$

Satisfiability Problem

An example of propositional conjunctive normal form (CNF) is like this:

$$(p \vee q \vee s) \wedge (\overline{q} \vee r) \wedge (\overline{p} \vee r) \wedge (\overline{r} \vee s) \wedge (\overline{p} \vee \overline{s} \vee \overline{q})$$

Satisfiability Problem

Given a CNF formula, is there a truth assignment that satisfies it?

In other words, is there a assignment for the set of propositional variable in the CNF, such that the value of the formula is **true**.

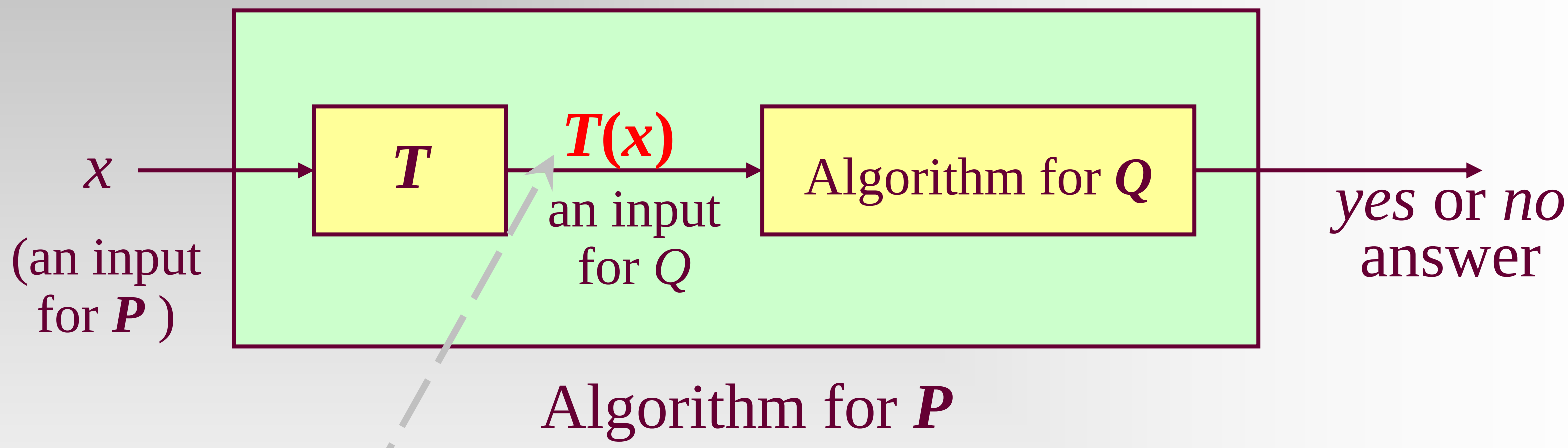
```
void nondetSat(E, n)
    boolean p[ ];
    for int i =1 to n do
        p[i ]= genCertif(true, false);
    if E(p[1], p[2], ..., p[n])=true
        then Output("yes");
```

So, the problem is in *NP*

Relation between *P* and *NP*

- An deterministic algorithm for a decision problem is a special case of a nondeterministic algorithm, which means: $P \subseteq NP$
  - The deterministic algorithm is looked as the phase 2 of a nondeterministic one, which always ignore the *s* the phase 1 has written.
- Intuition implies that *NP* is a much larger set than *P*.
  - The number of possible *s* is exponential in *n*.
  - No one problem in *NP* has been proved not in *P*.

Solving a Problem Indirectly



The correct answer for ***P*** on x is *yes* **if and only if** the correct answer for ***Q*** on *T*(*x*) is *yes*.

Polynomial Reduction

- Let T be a function from the input set for a decision problem P into the input set for Q. T is a polynomial reduction from P to Q if:
  - T can be computed in polynomial bounded time
  - x is a *yes* input for $P \rightarrow T(x)$ is a *yes* input for Q
  - x is a *no* input for $P \rightarrow T(x)$ is a *no* input for Q

An example:

P: Given a sequence of Boolean values, does at least one of them have the value true?

Q: Given a sequence of integers, is the maximum of them positive?

$T(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$, where: $y_i=1$ if x_i=true, and $y_i=0$ if x_i=false

Relation of Reducibility

- Problem P is **polynomially reducible** to Q if there exists a polynomial reduction from P to Q, denoted as: $\boldsymbol{P \leq_P Q}$
- If $P \leq_P Q$ and Q is in ***P***, then P is in ***P***
  - The complexity of P is the sum of T, with the input size n, and Q, with the input size $p(n)$, where p is the polynomial bound on T,
  - So, the total cost is: $p(n)+q(p(n))$, where q is the polynomial bound on Q.

(If $P \leq_P Q$, then Q is at least as "hard" to solve as P)

NP-complete Problems

- A problem *Q* is ***NP*-hard** if every problem *P* in *NP* is reducible to *Q*, that is $P \leq_P Q$.

  (which means that *Q* is at least as hard as any problem in *NP*)

- A problem *Q* is ***NP*-complete** if it is in *NP* and is ***NP*-hard**.

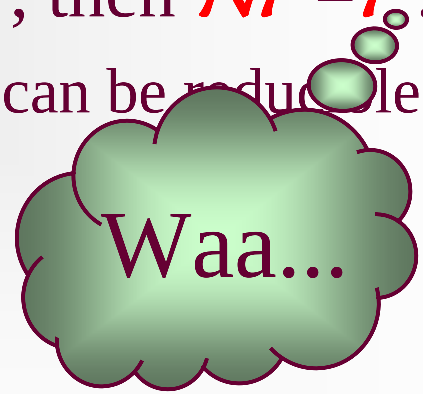
  (which means that *Q* is at most as hard as to be solved by a polynomially bounded nondeterministic algorithm)

An Example of *NP*-**hard** problem

- Halt problem: Given an arbitrary deterministic algorithm *A* and an input *I*, does *A* with input *I* ever terminate?
  - A well-known **undecidable** problem, of course not in ***NP***.
  - Satisfiability problem is reducible to it.
    - Construct an algorithm *A* whose input is a propositional formula *X*. If *X* has *n* variables then *A* tries out all 2^n possible truth assignments and verifies if *X* is satisfiable. If it is satisfiable then *A* stops. Otherwise, *A* enters an infinite loop.
    - So, *A* halts on *X* iff. *X* is satisfiable.

P and *NP*: Revisited

- Intuition implies that *NP* is a much larger set than *P*.
  - The number of possible *s* is exponential in *n*.
  - No one problem in *NP* has been proved not in *P*.
- If any *NP*-completed problem is in *P*, then *NP*=*P*.
  - Which means that every problems in *NP* can be reducible to a problem in *P*!
  - Much more questionable!!



First Known *NP*-Complete Problem

- Cook's theorem:
  - **The satisfiability problem is *NP*-complete.**
- Reduction as tool for proving *NP*-complete
  - Since *CNF-SAT* is known to be *NP*-hard, then all the problems, to which *CNF-SAT* is reducible, are also *NP*-hard. So, the formidable task of proving *NP*-complete is transformed into relatively easy task of proving of being in ***NP***.

Procedure for NP-Completeness

- Knowledge: *P* is *NP*-completeness
- Task: to prove that *Q* is *NP*-complete
- Approach: to reduce *P* to *Q*
  - For any $R \in NP$, $R \leq_P P$
  - Show $P \leq_P Q$
  - Then $R \leq_P Q$, by transitivity of reductions
  - Done. *Q* is *NP*-complete

Proof of Cook's Theorem

COOK, S. 1971.

The complexity of theorem-proving procedures.

In

Conference Record of

3rd Annual ACM Symposium on Theory of Computing.

ACM New York, pp. 151–158.

Stephen Arthur Cook: b.1939 in Buffalo, NY. Ph.D of Harvard. Professor of Toronto Univ. 1982 Turing Award winner. The Turing Award lecture: "An Overview of Computational Complexity", CACM, June 1983, pp.400-8

Home Assignment

- pp.600-
  - 13.1
  - 13.3
  - 13.4
  - 13.6