

# Supplementary Material: Sparse Learning for Large-scale and High-dimensional Data

Lijun Zhang<sup>1</sup>, Tianbao Yang<sup>2</sup>, Rong Jin<sup>3</sup>, and Zhi-Hua Zhou<sup>1</sup>

<sup>1</sup> National Key Laboratory for Novel Software Technology, Nanjing University,  
Nanjing 210023, China

{zhanglj, zhoush}@lamda.nju.edu.cn

<sup>2</sup> Department of Computer Science, The University of Iowa, Iowa City 52242, USA  
tianbao-yang@uiowa.edu

<sup>3</sup> Alibaba Group, Seattle, USA  
jinrong.jr@alibaba-inc.com

## A Proofs of Lemmas for Supporting Theorem 1

### A.1 Proof of Lemma 4

Let  $\Omega_w$  include the subset of non-zeros entries in  $\mathbf{w}_*$  and  $\bar{\Omega}_w = [d] \setminus \Omega_w$ . Define

$$\begin{aligned}\mathcal{G}(\mathbf{w}) &= g(\mathbf{w}) + \max_{\lambda \in \Delta} -h(\lambda) - \mathbf{w}^\top A \lambda, \\ \hat{\mathcal{G}}(\mathbf{w}) &= g(\mathbf{w}) + \gamma_w \|\mathbf{w}\|_1 + \max_{\lambda \in \Delta} -h(\lambda) - \mathbf{w}^\top \hat{A} R^\top \lambda - \gamma_w \|\lambda\|_1.\end{aligned}$$

Let  $\mathbf{v} \in \partial \|\mathbf{w}_*\|_1$  be any subgradient of  $\|\cdot\|_1$  at  $\mathbf{w}_*$ . Then, we have

$$\mathbf{u} = \nabla g(\mathbf{w}_*) - ARR^\top \tilde{\lambda} + \gamma_w \mathbf{v} \in \partial \hat{\mathcal{G}}(\mathbf{w}_*).^4$$

Using the fact that  $\hat{\mathbf{w}}$  minimizes  $\hat{\mathcal{G}}(\cdot)$  over the domain  $\Omega$  and  $g(\cdot)$  is  $\alpha$ -strongly convex, we have

$$\begin{aligned}0 &\geq \hat{\mathcal{G}}(\hat{\mathbf{w}}) - \hat{\mathcal{G}}(\mathbf{w}_*) \geq \langle \hat{\mathbf{w}} - \mathbf{w}_*, \mathbf{u} \rangle + \frac{\alpha}{2} \|\hat{\mathbf{w}} - \mathbf{w}_*\|_2^2 \\ &= \langle \hat{\mathbf{w}} - \mathbf{w}_*, \nabla g(\mathbf{w}_*) - ARR^\top \tilde{\lambda} + \gamma_w \mathbf{v} \rangle + \frac{\alpha}{2} \|\hat{\mathbf{w}} - \mathbf{w}_*\|_2^2.\end{aligned}\tag{21}$$

By setting  $v_i = \text{sign}(\hat{w}_i)$ ,  $\forall i \in \bar{\Omega}_w$ , we have  $\langle \hat{\mathbf{w}}_{\bar{\Omega}_w}, \mathbf{v}_{\bar{\Omega}_w} \rangle = \|\hat{\mathbf{w}}_{\bar{\Omega}_w}\|_1$ . As a result,

$$\langle \hat{\mathbf{w}} - \mathbf{w}_*, \mathbf{v} \rangle = \langle \hat{\mathbf{w}}_{\bar{\Omega}_w}, \mathbf{v}_{\bar{\Omega}_w} \rangle + \langle \hat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*, \mathbf{v}_{\Omega_w} \rangle \geq \|\hat{\mathbf{w}}_{\bar{\Omega}_w}\|_1 - \|\hat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1.\tag{22}$$

Combining (21) with (22), we have

$$\left\langle \hat{\mathbf{w}} - \mathbf{w}_*, \nabla g(\mathbf{w}_*) - ARR^\top \tilde{\lambda} \right\rangle + \frac{\alpha}{2} \|\hat{\mathbf{w}} - \mathbf{w}_*\|_2^2 + \gamma_w \|\hat{\mathbf{w}}_{\bar{\Omega}_w}\|_1 \leq \gamma_w \|\hat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1.\tag{23}$$

<sup>4</sup> In the case that  $g(\cdot)$  is non-smooth,  $\nabla g(\mathbf{w}_*)$  refers to a subgradient of  $g(\cdot)$  at  $\mathbf{w}_*$ . In particular, we choose the subgradient that satisfies (24).

From the fact that  $\mathbf{w}_*$  minimizes  $\mathcal{G}(\cdot)$  over the domain  $\Omega$ , we have

$$\langle \nabla \mathcal{G}(\mathbf{w}_*), \mathbf{w} - \mathbf{w}_* \rangle = \langle \nabla g(\mathbf{w}_*) - A\boldsymbol{\lambda}_*, \mathbf{w} - \mathbf{w}_* \rangle \geq 0, \quad \forall \mathbf{w} \in \Omega. \quad (24)$$

Then,

$$\begin{aligned} & \left\langle \widehat{\mathbf{w}} - \mathbf{w}_*, \nabla g(\mathbf{w}_*) - ARR^\top \tilde{\boldsymbol{\lambda}} \right\rangle \\ &= \langle \widehat{\mathbf{w}} - \mathbf{w}_*, \nabla g(\mathbf{w}_*) - A\boldsymbol{\lambda}_* \rangle + \langle \widehat{\mathbf{w}} - \mathbf{w}_*, A(I - RR^\top)\boldsymbol{\lambda}_* \rangle + \left\langle \widehat{\mathbf{w}} - \mathbf{w}_*, ARR^\top(\boldsymbol{\lambda}_* - \tilde{\boldsymbol{\lambda}}) \right\rangle \\ &\stackrel{(24)}{\geq} -\|\widehat{\mathbf{w}} - \mathbf{w}_*\|_1 \left( \|A(I - RR^\top)\boldsymbol{\lambda}_*\|_\infty + \|ARR^\top(\boldsymbol{\lambda}_* - \tilde{\boldsymbol{\lambda}})\|_\infty \right) \\ &\stackrel{(14)}{=} -\rho_w \|\widehat{\mathbf{w}} - \mathbf{w}_*\|_1 = -\rho_w (\|\widehat{\mathbf{w}}_{\bar{\Omega}_w}\|_1 + \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1). \end{aligned} \quad (25)$$

From (23) and (25), we have

$$\frac{\alpha}{2} \|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2^2 + (\gamma_w - \rho_w) \|\widehat{\mathbf{w}}_{\bar{\Omega}_w}\|_1 \leq (\gamma_w + \rho_w) \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1.$$

Since  $\gamma_w \geq 2\rho_w$ , we have

$$\frac{\alpha}{2} \|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2^2 + \frac{\gamma_w}{2} \|\widehat{\mathbf{w}}_{\bar{\Omega}_w}\|_1 \leq \frac{3\gamma_w}{2} \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1.$$

And thus,

$$\begin{aligned} \frac{\alpha}{2} \|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2^2 &\leq \frac{3\gamma_w}{2} \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1 \leq \frac{3\gamma_w \sqrt{s_w}}{2} \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_2 \Rightarrow \|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2 \leq \frac{3\gamma_w \sqrt{s_w}}{\alpha} \\ \frac{\alpha}{2s_w} \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1^2 &\leq \frac{\alpha}{2} \|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2^2 \leq \frac{3\gamma_w}{2} \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1 \Rightarrow \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1 \leq \frac{3\gamma_w s_w}{\alpha} \\ \frac{\gamma_w}{2} \|\widehat{\mathbf{w}}_{\bar{\Omega}_w}\|_1 &\leq \frac{3\gamma_w}{2} \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1 \Rightarrow \|\widehat{\mathbf{w}}_{\bar{\Omega}_w}\|_1 \leq 3\|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1 \Rightarrow \|\widehat{\mathbf{w}} - \mathbf{w}_*\|_1 \leq \frac{12\gamma_w s_w}{\alpha} \\ \frac{\|\widehat{\mathbf{w}} - \mathbf{w}_*\|_1}{\|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2} &= \frac{\|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1 + \|\widehat{\mathbf{w}}_{\bar{\Omega}_w}\|_1}{\|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2} \leq \frac{4\|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1}{\|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2} \leq \frac{4\sqrt{s_w} \|\widehat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_2}{\|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2} \leq 4\sqrt{s_w}. \end{aligned}$$

## A.2 Proof of Lemma 6

First, we assume  $\|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1$ . Following the proof of Corollary 2 in [3], we apply Property 1 to vectors  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u} - \mathbf{v}$ . Then, with a probability at least  $1 - 4\exp(-m\varepsilon^2/c)$ , we have

$$(1 - \varepsilon) \|\mathbf{u} + \mathbf{v}\|_2^2 \leq \|R^\top(\mathbf{u} + \mathbf{v})\|_2^2 \leq (1 + \varepsilon) \|\mathbf{u} + \mathbf{v}\|_2^2, \quad (26)$$

$$(1 - \varepsilon) \|\mathbf{u} - \mathbf{v}\|_2^2 \leq \|R^\top(\mathbf{u} - \mathbf{v})\|_2^2 \leq (1 + \varepsilon) \|\mathbf{u} - \mathbf{v}\|_2^2, \quad (27)$$

provided  $\varepsilon \leq 1/2$ . From (26) and (27), it is straightforward to show that

$$|\mathbf{u}^\top RR^\top \mathbf{v} - \mathbf{u}^\top \mathbf{v}| \leq \varepsilon.$$

Thus, with a probability at least  $1 - \delta$ , we have

$$|\mathbf{u}^\top RR^\top \mathbf{v} - \mathbf{u}^\top \mathbf{v}| \leq \sqrt{\frac{c}{m} \log \frac{4}{\delta}}$$

provided (10) holds.

We complete the proof by noticing

$$|\mathbf{u}^\top RR^\top \mathbf{v} - \mathbf{u}^\top \mathbf{v}| = \|\mathbf{u}\|_2 \|\mathbf{v}\|_2 \left| \frac{\mathbf{u}^\top}{\|\mathbf{u}\|_2} RR^\top \frac{\mathbf{v}}{\|\mathbf{v}\|_2} - \frac{\mathbf{u}^\top \mathbf{v}}{\|\mathbf{u}\|_2 \|\mathbf{v}\|_2} \right|.$$

### A.3 Proof of Lemma 7

First, we define

$$\mathcal{S}_{n,16s_\lambda} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 \leq 1, \|\mathbf{x}\|_0 \leq 16s_\lambda\}.$$

Using Lemma 3.1 from [25], we have  $\mathcal{K}_{n,16s_\lambda} \subset 2 \operatorname{conv}(\mathcal{S}_{n,16s_\lambda})$  and therefore

$$U_4 \leq 2 \sup_{\mathbf{z} \in \operatorname{conv}(\mathcal{S}_{n,16s_\lambda})} \|A(RR^\top - I)\mathbf{z}\|_\infty = 2 \underbrace{\sup_{\mathbf{z} \in \mathcal{S}_{n,16s_\lambda}} \|A(RR^\top - I)\mathbf{z}\|_\infty}_{:=\theta} \quad (28)$$

where the last equality follows from the fact that the maximum of a convex function over a convex set generally occurs at some extreme point of the set [27].

Let  $\mathcal{S}_{n,s}(\epsilon)$  be a proper  $\epsilon$ -net for  $\mathcal{S}_{n,s}$  with the smallest cardinality, and  $|\mathcal{S}_{n,s}(\epsilon)|$  be the covering number for  $\mathcal{S}_{n,s}$ . We have the following lemma for bounding  $|\mathcal{S}_{n,s}(\epsilon)|$ .

**Lemma 8** [25, Lemma 3.3] *For  $\epsilon \in (0, 1)$  and  $s \leq n$ , we have*

$$\log |\mathcal{S}_{n,s}(\epsilon)| \leq s \log \left( \frac{9n}{\epsilon s} \right).$$

Let  $\mathcal{S}_{n,16s_\lambda}(\epsilon)$  be a  $\epsilon$ -net of  $\mathcal{S}_{n,16s_\lambda}$  with smallest cardinality. With the help of  $\mathcal{S}_{n,16s_\lambda}(\epsilon)$ , we define a discretized version of  $\theta$  in (28) as

$$\theta(\epsilon) = \sup \{ \|A(RR^\top - I)\mathbf{z}\|_\infty : \mathbf{z} \in \mathcal{S}_{n,16s_\lambda}(\epsilon) \}.$$

The following lemma relates  $\theta$  with  $\theta(\epsilon)$ .

**Lemma 9** [17, Lemma 9.2] *For  $\epsilon \in (0, 1/\sqrt{2})$ , we have*

$$\theta \leq \frac{\theta(\epsilon)}{1 - \sqrt{2}\epsilon}.$$

By choosing  $\epsilon = 1/2$ , we have  $\theta \leq (2 + \sqrt{2})\theta(1/2)$ . Combining with (28), we obtain

$$U_4 \leq 2(2 + \sqrt{2}) \underbrace{\sup \{ \|A(RR^\top - I)\mathbf{z}\|_\infty : \mathbf{z} \in \mathcal{S}_{n,16s_\lambda}(1/2) \}}_{\theta(1/2)}$$

Furthermore, Lemma 8 implies

$$\log |\mathcal{S}_{n,16s_\lambda}(1/2)| \leq 16s_\lambda \log \left( \frac{9n}{8s_\lambda} \right).$$

We proceed by providing an upper bound for  $\theta(1/2)$ . Following the arguments for bounding  $U_1$  in the proof of Lemma 5, we have with a probability at least  $1 - \delta$ ,

$$\|A(RR^\top - I)\mathbf{z}\|_\infty \leq \sqrt{\frac{c}{m} \log \frac{4d}{\delta}}$$

for each  $\mathbf{z} \in \mathcal{S}_{n,16s_\lambda}(1/2)$ . We complete the proof by taking the union bound over all  $\mathbf{z} \in \mathcal{S}_{n,16s_\lambda}(1/2)$ .

## B Proof of Theorem 2

The analysis here is similar to that for Lemma 1. Recall that in the proof of Theorem 1, we have proved that

$$\gamma_\lambda \geq 2\|A^\top \mathbf{w}_*\|_2 \sqrt{\frac{c}{m} \log \frac{4n}{\delta}} \geq 2\|(RR^\top - I)A^\top \mathbf{w}_*\|_\infty \quad (29)$$

holds with a probability at least  $1 - \delta$ .

Define

$$\widehat{\mathcal{L}}(\boldsymbol{\lambda}) = -h(\boldsymbol{\lambda}) - \widehat{\mathbf{w}}^\top \widehat{A}R^\top \boldsymbol{\lambda} - \gamma_\lambda \|\boldsymbol{\lambda}\|_1.$$

Using the fact that  $\widehat{\boldsymbol{\lambda}}$  maximizes  $\widehat{\mathcal{L}}(\cdot)$  over the domain  $\Delta$  and  $h(\cdot)$  is  $\beta$ -strongly convex, we have

$$\left\langle \widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*, \nabla h(\boldsymbol{\lambda}_*) + RR^\top A^\top \widehat{\mathbf{w}} \right\rangle + \frac{\beta}{2} \|\boldsymbol{\lambda}_* - \widehat{\boldsymbol{\lambda}}\|_2^2 + \gamma_\lambda \|\widehat{\boldsymbol{\lambda}}_{\Omega_\lambda}\|_1 \leq \gamma_\lambda \|\widehat{\boldsymbol{\lambda}}_{\Omega_\lambda} - \boldsymbol{\lambda}_*\|_1. \quad (30)$$

On the other hand, we have

$$\begin{aligned} & \left\langle \widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*, \nabla h(\boldsymbol{\lambda}_*) + RR^\top A^\top \widehat{\mathbf{w}} \right\rangle \\ &= \left\langle \widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*, \nabla h(\boldsymbol{\lambda}_*) + A^\top \mathbf{w}_* \right\rangle + \left\langle \widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*, (RR^\top - I)A^\top \mathbf{w}_* \right\rangle + \left\langle \widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*, RR^\top A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*) \right\rangle \\ &\stackrel{(18)}{\geq} -\|\widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_1 \|(RR^\top - I)A^\top \mathbf{w}_*\|_\infty - \|\widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_2 \|RR^\top A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*)\|_2 \\ &\stackrel{(29)}{\geq} -\frac{\gamma_\lambda}{2} \|\widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_1 - \|\widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_2 \|RR^\top A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*)\|_2. \end{aligned} \quad (31)$$

From (30) and (31), we have

$$\begin{aligned}
& \frac{\beta}{2} \|\boldsymbol{\lambda}_* - \widehat{\boldsymbol{\lambda}}\|_2^2 + \frac{\gamma\lambda}{2} \|\widehat{\boldsymbol{\lambda}}_{\Omega_\lambda}\|_1 \\
& \leq \frac{3\gamma\lambda}{2} \|\widetilde{\boldsymbol{\lambda}}_{\Omega_\lambda} - \boldsymbol{\lambda}_*\|_1 + \|\widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_2 \|RR^\top A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*)\|_2 \\
& \leq \frac{3\gamma\lambda\sqrt{s_\lambda}}{2} \|\widetilde{\boldsymbol{\lambda}}_{\Omega_\lambda} - \boldsymbol{\lambda}_*\|_2 + \|\widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_2 \|RR^\top A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*)\|_2 \\
& \leq \|\widehat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_2 \left( \frac{3\gamma\lambda\sqrt{s_\lambda}}{2} + \|RR^\top A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*)\|_2 \right)
\end{aligned}$$

which implies

$$\begin{aligned}
& \|\boldsymbol{\lambda}_* - \widehat{\boldsymbol{\lambda}}\|_2 \\
& \leq \frac{2}{\beta} \left( \frac{3\gamma\lambda\sqrt{s_\lambda}}{2} + \|RR^\top A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*)\|_2 \right) \\
& \leq \frac{2}{\beta} \left( \frac{3\gamma\lambda\sqrt{s_\lambda}}{2} + \|A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*)\|_2 + \|(RR^\top - I)A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*)\|_2 \right) \\
& \leq \frac{2}{\beta} \left( \frac{3\gamma\lambda\sqrt{s_\lambda}}{2} + (1 + \|RR^\top - I\|_2) \|A^\top (\widehat{\mathbf{w}} - \mathbf{w}_*)\|_2 \right).
\end{aligned}$$

## C Proof of Theorem 4

The proof is almost identical to that of Theorem 1. We just need to replace Lemmas 1 and 4 with the following ones.

**Lemma 10** *Denote*

$$\rho_\lambda = \|(RR^\top - I)A^\top \mathbf{w}_*\|_\infty + \varsigma. \quad (32)$$

*By choosing  $\gamma_\lambda \geq 2\rho_\lambda$ , we have*

$$\|\widetilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_2 \leq \frac{3\gamma_\lambda\sqrt{s_\lambda}}{\beta}, \quad \|\widetilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_1 \leq \frac{12\gamma_\lambda s_\lambda}{\beta}, \quad \text{and} \quad \frac{\|\widetilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_1}{\|\widetilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_2} \leq 4\sqrt{s_\lambda}.$$

**Lemma 11** *Denote*

$$\rho_w = \|A(I - RR^\top)\boldsymbol{\lambda}_*\|_\infty + \|ARR^\top(\boldsymbol{\lambda}_* - \widetilde{\boldsymbol{\lambda}})\|_\infty + \varsigma. \quad (33)$$

*By choosing  $\gamma_w \geq 2\rho_w$ , we have*

$$\|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2 \leq \frac{3\gamma_w\sqrt{s_w}}{\alpha}, \quad \|\widehat{\mathbf{w}} - \mathbf{w}_*\|_1 \leq \frac{12\gamma_w s_w}{\alpha}, \quad \text{and} \quad \frac{\|\widehat{\mathbf{w}} - \mathbf{w}_*\|_1}{\|\widehat{\mathbf{w}} - \mathbf{w}_*\|_2} \leq 4\sqrt{s_w}.$$

### C.1 Proof of Lemma 10

From the assumption, we have

$$\begin{aligned}
& \left\langle \tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*, \nabla h(\boldsymbol{\lambda}_*) + RR^\top A^\top \mathbf{w}_* \right\rangle \\
&= \left\langle \tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*, \nabla h(\boldsymbol{\lambda}_*) + A^\top \mathbf{w}_* \right\rangle + \left\langle \tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*, (RR^\top - I)A^\top \mathbf{w}_* \right\rangle \\
&\stackrel{(12)}{\geq} -\|\tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_1 \left( \|(RR^\top - I)A^\top \mathbf{w}_*\|_\infty + \varsigma \right) \\
&\stackrel{(32)}{=} -\rho_\lambda \|\tilde{\boldsymbol{\lambda}} - \boldsymbol{\lambda}_*\|_1 = -\rho_\lambda \left( \|\tilde{\boldsymbol{\lambda}}_{\bar{\Omega}_\lambda}\|_1 + \|\tilde{\boldsymbol{\lambda}}_{\Omega_\lambda} - \boldsymbol{\lambda}_*\|_1 \right).
\end{aligned}$$

Substituting the above inequality into (17), and the rest proof is identical to that of Lemma 1.

### C.2 Proof of Lemma 11

Similarly, we have

$$\begin{aligned}
& \left\langle \hat{\mathbf{w}} - \mathbf{w}_*, \nabla g(\mathbf{w}_*) - ARR^\top \tilde{\boldsymbol{\lambda}} \right\rangle \\
&= \left\langle \hat{\mathbf{w}} - \mathbf{w}_*, \nabla g(\mathbf{w}_*) - A\boldsymbol{\lambda}_* \right\rangle + \left\langle \hat{\mathbf{w}} - \mathbf{w}_*, A(I - RR^\top)\boldsymbol{\lambda}_* \right\rangle + \left\langle \hat{\mathbf{w}} - \mathbf{w}_*, ARR^\top(\boldsymbol{\lambda}_* - \tilde{\boldsymbol{\lambda}}) \right\rangle \\
&\stackrel{(11)}{\geq} -\|\hat{\mathbf{w}} - \mathbf{w}_*\|_1 \left( \|A(I - RR^\top)\boldsymbol{\lambda}_*\|_\infty + \|ARR^\top(\boldsymbol{\lambda}_* - \tilde{\boldsymbol{\lambda}})\|_\infty + \varsigma \right) \\
&\stackrel{(33)}{=} -\rho_w \|\hat{\mathbf{w}} - \mathbf{w}_*\|_1 = -\rho_w \left( \|\hat{\mathbf{w}}_{\bar{\Omega}_w}\|_1 + \|\hat{\mathbf{w}}_{\Omega_w} - \mathbf{w}_*\|_1 \right).
\end{aligned}$$

Substituting the above inequality into (23), and the rest proof is identical to that of Lemma 4.