Supplementary Material: Online Bandit Learning for a Special Class of Non-convex Losses

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Proof of Proposition 1

The proof is similar to that of Lemma 4 in (Zhang, Yi, and Jin 2014). First, we have

$$\begin{aligned} & \mathbf{u}^\top \mathrm{E}_{t-1}\left[-\frac{Z_t c_t}{\|\mathbf{v}_t\|_2}\mathbf{v}_t\right] \\ \overset{(1)}{=} & -\eta \mathrm{E}_{t-1}\left[f\left(\mathbf{u}^\top \frac{\mathbf{v}_t}{\|\mathbf{v}_t\|_2}\right)\mathbf{u}^\top \frac{\mathbf{v}_t}{\|\mathbf{v}_t\|_2}\right] \\ \overset{(3),(5)}{=} & \eta\gamma. \end{aligned}$$

Then, consider any vector $\bar{\mathbf{u}} \perp \mathbf{u}$. We have

$$\begin{aligned} & \bar{\mathbf{u}}^\top \mathrm{E}_{t-1}\left[-\frac{Z_t c_t}{\|\mathbf{v}_t\|_2}\mathbf{v}_t\right] \\ \overset{(1)}{=} & -\eta \mathrm{E}_{t-1}\left[f\left(\mathbf{u}^\top \frac{\mathbf{v}_t}{\|\mathbf{v}_t\|_2}\right)\bar{\mathbf{u}}^\top \frac{\mathbf{v}_t}{\|\mathbf{v}_t\|_2}\right] \\ = & -\eta \mathrm{E}_{t-1}\left[f\left(\frac{\alpha_1}{\sqrt{\alpha_1^2+\cdots+\alpha_d^2}}\right)\frac{\|\bar{\mathbf{u}}\|_2\alpha_2}{\sqrt{\alpha_1^2+\cdots+\alpha_d^2}}\right] \end{aligned}$$

where $\alpha_1, \ldots, \alpha_n$ are n independent standard Gaussian variables. Notice that $f\left(\frac{\alpha_1}{\sqrt{\alpha_1^2+\cdots+\alpha_d^2}}\right)\frac{\|\bar{\mathbf{u}}\|_2\alpha_2}{\sqrt{\alpha_1^2+\cdots+\alpha_d^2}}$ is an odd function of α_2, so its expectation with respect to α_2 must be 0. Thus, we have

$$\bar{\mathbf{u}}^\top \mathrm{E}_{t-1}\left[-\frac{Z_t c_t}{\|\mathbf{v}_t\|_2}\mathbf{v}_t\right] = 0.$$

Then, it is easy to prove Proposition 1 by contradiction.

Proof of Theorem 3

The following lemma extends Lemma 2 to the general case when each $\mathbf{u}_t$ is a different vector.

Lemma 4. *We have*

$$f(\mathbf{x}_t^\top \mathbf{u}_t) - \min_{\|\mathbf{x}\|\leq 1} f(\mathbf{x}^\top \mathbf{u}_t) \leq \begin{cases} 2B, & \text{if } t=1; \\ \frac{2L}{\gamma\eta\rho_T(t-1)}\left\|\sum_{i=1}^{t-1}\delta_i\right\|_2 + 2L\|\mathbf{u}_t - \bar{\mathbf{u}}_{t-1}\|_2, & \text{otherwise.} \end{cases}$$

where $\rho_T = \min_{t\in[T]} \|\bar{\mathbf{u}}_t\|_2$.

The rest proof is the same as that for Theorem 1.

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Proof of Lemma 4

When $t=1$, it is clear that

$$f(\mathbf{x}_1^\top \mathbf{u}_1) - \min_{\|\mathbf{x}\|\leq 1} f(\mathbf{x}^\top \mathbf{u}_1) \overset{(2)}{\leq} 2B.$$

In the following, we discuss the case when $t \geq 2$. Notice that in this setting, (4) becomes

$$\mathrm{E}_{t-1}\left[-\frac{Z_t c_t}{\|\mathbf{v}_t\|_2}\mathbf{v}_t\right] = \gamma\eta\mathbf{u}_t,\ t=1,\ldots,T \tag{16}$$

and the vector-valued martingale-difference sequence is

$$\delta_i = -\frac{Z_i c_i}{\|\mathbf{v}_i\|_2}\mathbf{v}_i - \gamma\eta\mathbf{u}_i,\ i=1,\ldots,T. \tag{17}$$

Following the same analysis for Lemma 2, we have

$$\begin{aligned} & f(\mathbf{x}_t^\top \mathbf{u}_t) - \min_{\|\mathbf{x}\|\leq 1} f(\mathbf{x}^\top \mathbf{u}_t) = f(\mathbf{x}_t^\top \mathbf{u}_t) - f(\mathbf{u}_t^\top \mathbf{u}_t) \\ \leq & L|\mathbf{x}_t^\top \mathbf{u}_t - \mathbf{u}_t^\top \mathbf{u}_t| \leq L\|\mathbf{x}_t - \mathbf{u}_t\|_2 = L\left\|\frac{\mathbf{w}_t}{\|\mathbf{w}_t\|} - \mathbf{u}_t\right\|_2 \\ \leq & L\left\|\frac{\mathbf{w}_t}{\|\mathbf{w}_t\|} - \frac{\bar{\mathbf{u}}_{t-1}}{\|\bar{\mathbf{u}}_{t-1}\|_2}\right\|_2 + L\left\|\mathbf{u}_t - \frac{\bar{\mathbf{u}}_{t-1}}{\|\bar{\mathbf{u}}_{t-1}\|_2}\right\|_2. \end{aligned} \tag{18}$$

According to the procedure in Algorithm 1, we have

$$\begin{aligned} \mathbf{w}_t = & \sum_{i=1}^{t-1} -\frac{Z_i c_i}{\|\mathbf{v}_i\|_2}\mathbf{v}_i \\ \overset{(17)}{=} & \gamma\eta\sum_{i=1}^{t-1}\mathbf{u}_i + \sum_{i=1}^{t-1}\delta_i \\ = & \gamma\eta(t-1)\bar{\mathbf{u}}_{t-1} + \sum_{i=1}^{t-1}\delta_i. \end{aligned}$$

Then, we have

$$\begin{aligned} & \left\|\frac{\mathbf{w}_t}{\gamma\eta(t-1)\|\bar{\mathbf{u}}_{t-1}\|_2} - \frac{\bar{\mathbf{u}}_{t-1}}{\|\bar{\mathbf{u}}_{t-1}\|_2}\right\|_2 \\ \leq & \frac{1}{\gamma\eta(t-1)\|\bar{\mathbf{u}}_{t-1}\|_2}\left\|\sum_{i=1}^{t-1}\delta_i\right\|_2. \end{aligned}$$

Following a simple geometric argument, we have

$$\left\| \frac{\mathbf{w}_t}{\|\mathbf{w}_t\|} - \frac{\bar{\mathbf{u}}_{t-1}}{\|\bar{\mathbf{u}}_{t-1}\|_2} \right\|_2 \leq \frac{2}{\gamma\eta(t-1)\|\bar{\mathbf{u}}_{t-1}\|_2} \left\| \sum_{i=1}^{t-1} \delta_i \right\|_2,$$

implying

$$\begin{aligned} & f(\mathbf{x}_t^\top \mathbf{u}) - \min_{\|\mathbf{x}\|\leq 1} f(\mathbf{x}^\top \mathbf{u}) \\ \stackrel{(18)}{\leq} & \frac{2L}{\gamma\eta(t-1)\|\bar{\mathbf{u}}_{t-1}\|_2} \left\| \sum_{i=1}^{t-1} \delta_i \right\|_2 + L \left\| \mathbf{u}_t - \frac{\bar{\mathbf{u}}_{t-1}}{\|\bar{\mathbf{u}}_{t-1}\|_2} \right\|_2 \\ \leq & \frac{2L}{\gamma\eta\rho_T(t-1)} \left\| \sum_{i=1}^{t-1} \delta_i \right\|_2 + 2L\|\mathbf{u}_t - \bar{\mathbf{u}}_{t-1}\|_2. \end{aligned}$$

Proof of (7)

We provide the proof because our definition of $\bar{\mathbf{u}}_t$ is slightly different from the one in (Hazan and Kale 2010).

First, we have

$$\begin{aligned} & \sum_{t=2}^{T} \|\mathbf{u}_t - \bar{\mathbf{u}}_{t-1}\|_2^2 - \sum_{t=2}^{T} \|\mathbf{u}_t - \bar{\mathbf{u}}_t\|_2^2 \\ \leq & \sum_{t=2}^{T} 2\langle \mathbf{u}_t - \bar{\mathbf{u}}_{t-1}, \bar{\mathbf{u}}_t - \bar{\mathbf{u}}_{t-1} \rangle \\ = & \sum_{t=2}^{T} 2 \left\langle \mathbf{u}_t - \bar{\mathbf{u}}_{t-1}, \frac{(t-1)\bar{\mathbf{u}}_{t-1} + \mathbf{u}_t}{t} - \bar{\mathbf{u}}_{t-1} \right\rangle \\ = & 2\sum_{t=2}^{T} \frac{\|\mathbf{u}_t - \bar{\mathbf{u}}_{t-1}\|_2^2}{t} \leq 4 \sum_{t=2}^{T} \frac{\|\mathbf{u}_t - \bar{\mathbf{u}}_{t-1}\|_2}{t} \end{aligned} \tag{19}$$

where the first inequality is due to the property of convexity and the second inequality comes from our assumption that $\|\mathbf{u}_t\|_2 = 1, \forall t \in [T]$. We then bound $\sum_{t=2}^{T} \frac{\|\mathbf{u}_t - \bar{\mathbf{u}}_{t-1}\|_2}{t}$ as follows.

$$\begin{aligned} & \sum_{t=2}^{T} \frac{\|\mathbf{u}_t - \bar{\mathbf{u}}_{t-1}\|_2}{t} = \sum_{t=2}^{T} \frac{1}{t} \left\| \mathbf{u}_t - \frac{1}{t-1} \sum_{i=1}^{t-1} \mathbf{u}_i \right\|_2 \\ = & \sum_{t=2}^{T} \frac{1}{t} \left\| (\mathbf{u}_t - \bar{\mathbf{u}}_T) - \frac{1}{t-1} \sum_{i=1}^{t-1} (\mathbf{u}_i - \bar{\mathbf{u}}_T) \right\|_2 \\ \leq & \sum_{t=2}^{T} \frac{1}{t} \|\mathbf{u}_t - \bar{\mathbf{u}}_T\|_2 + \sum_{t=1}^{T-1} \|\mathbf{u}_t - \bar{\mathbf{u}}_T\|_2 \sum_{i=t}^{T-1} \frac{1}{i(i+1)} \\ \leq & \sum_{t=2}^{T} \frac{1}{t} \|\mathbf{u}_t - \bar{\mathbf{u}}_T\|_2 + \sum_{t=1}^{T-1} \frac{1}{t} \|\mathbf{u}_t - \bar{\mathbf{u}}_T\|_2 \\ \leq & 2\sum_{t=1}^{T} \frac{1}{t} \|\mathbf{u}_t - \bar{\mathbf{u}}_T\|_2 \\ \leq & 2\sqrt{\sum_{t=1}^{T} \|\mathbf{u}_t - \bar{\mathbf{u}}_T\|_2^2} \sqrt{\sum_{t=1}^{T} \frac{1}{t^2}} \leq 3\sqrt{\sum_{t=1}^{T} \|\mathbf{u}_t - \bar{\mathbf{u}}_T\|_2^2}. \end{aligned} \tag{20}$$

Combining (19) with (20), we have

$$\begin{aligned} & \sum_{t=2}^{T} \|\mathbf{u}_t - \bar{\mathbf{u}}_{t-1}\|_2^2 \\ \leq & \sum_{t=2}^{T} \|\mathbf{u}_t - \bar{\mathbf{u}}_t\|_2^2 + 12\sqrt{\sum_{i=1}^{T} \|\mathbf{u}_t - \bar{\mathbf{u}}_T\|_2^2}. \end{aligned} \tag{21}$$

According to (Cesa-Bianchi and Lugosi 2006, Lemma 3.1), we have

$$\sum_{t=1}^{T} \|\mathbf{u}_t - \bar{\mathbf{u}}_t\|_2^2 \leq \sum_{t=1}^{T} \|\mathbf{u}_t - \bar{\mathbf{u}}_T\|_2^2. \tag{22}$$

We complete the proof by combining (21) with (22).

Multiplicative Chernoff Bound

Theorem 5. *(Angluin and Valiant 1979) Let $X_1, X_2, \ldots, X_n$ be independent binary random variables with $\Pr[X_i = 1] = p_i$. Denote $S = \sum_{i=1}^n X_i$ and $\mu = \mathrm{E}[S] = \sum_{i=1}^n p_i$. We have*

$$\Pr\left[S \leq (1-\epsilon)\mu\right] \leq \exp\left(-\frac{\epsilon^2}{2}\mu\right), \text{ for } 0 < \epsilon < 1,$$

$$\Pr\left[S \geq (1+\epsilon)\mu\right] \leq \exp\left(-\frac{\epsilon^2}{2+\epsilon}\mu\right), \text{ for } \epsilon > 0.$$

For the second bound, let $t = \frac{\epsilon^2}{2+\epsilon}\mu$, which implies $\epsilon = \frac{t+\sqrt{t^2+8\mu t}}{2\mu}$. Then, with a probability at least e^{-t}, we have

$$S \leq \left(1 + \frac{t+\sqrt{t^2+8\mu t}}{2\mu}\right)\mu \leq 2\mu + 2t.$$