Supplementary Material of Revisiting Smoothed Online Learning

Lijun Zhang\textsuperscript{1,2}, Wei Jiang\textsuperscript{1}, Shiyin Lu\textsuperscript{1}, Tianbao Yang\textsuperscript{3}
\textsuperscript{1}National Key Laboratory for Novel Software Technology, Nanjing University, Nanjing, China
\textsuperscript{2}Peng Cheng Laboratory, Shenzhen, Guangdong, China
\textsuperscript{3}Department of Computer Science, The University of Iowa, Iowa City, IA 52242, USA
\{zhanglj, jiangw, lusy\}@lamda.nju.edu.cn, tianbao-yang@uiowa.edu

A Analysis

In this section, we present the analysis of all the theorems.

A.1 Proof of Theorem \textsuperscript{1}

Recall that $x_t$ is the minimizer of $f_t(\cdot)$, which is $\alpha$-polyhedral. When $t \geq 2$, we have

\[ f_t(x_t) + \| x_t - x_{t-1} \| \leq f_t(x_t) + \| x_t - u_t \| + \| u_t - u_{t-1} \| + \| u_{t-1} - x_{t-1} \| \]

\[ \leq f_t(x_t) + \frac{1}{\alpha} (f_t(u_t) - f_t(x_t)) + \frac{1}{\alpha} (f_{t-1}(u_{t-1}) - f_{t-1}(x_{t-1})) + \| u_t - u_{t-1} \|. \]

For $t = 1$, we have

\[ f_1(x_1) + \| x_1 - x_0 \| \leq f_1(x_1) + \| x_1 - u_1 \| + \| u_1 - u_0 \| + \| u_0 - x_0 \| \]

\[ = f_1(x_1) + \| x_1 - u_1 \| + \| u_1 - u_0 \| \]

\[ \leq f_1(x_1) + \frac{1}{\alpha} (f_1(u_1) - f_1(x_1)) + \| u_1 - u_0 \|. \]

Summing over all the iterations, we have

\[
\sum_{t=1}^{T} \left( f_t(x_t) + \| x_t - x_{t-1} \| \right) \\
\leq \sum_{t=1}^{T} f_t(x_t) + \frac{1}{\alpha} \sum_{t=1}^{T} \left( f_t(u_t) - f_t(x_t) \right) + \frac{1}{\alpha} \sum_{t=2}^{T} \left( f_{t-1}(u_{t-1}) - f_{t-1}(x_{t-1}) \right) + \sum_{t=1}^{T} \| u_t - u_{t-1} \| \\
\leq \sum_{t=1}^{T} f_t(x_t) + \frac{2}{\alpha} \sum_{t=1}^{T} \left( f_t(u_t) - f_t(x_t) \right) + \sum_{t=1}^{T} \| u_t - u_{t-1} \| \\
= \frac{2}{\alpha} \sum_{t=1}^{T} f_t(u_t) + \sum_{t=1}^{T} \| u_t - u_{t-1} \| + \sum_{t=1}^{T} \left( 1 - \frac{2}{\alpha} \right) f_t(x_t). \]

(23)

where the second inequality follows from the fact that $f_T(x_T) \leq f_T(u_T)$.

Thus, if $\alpha \geq 2$, we have
\[
\sum_{t=1}^{T} \left( f_t(x_t) + \|x_t - x_{t-1}\| \right)
\leq \frac{2}{\alpha} \sum_{t=1}^{T} f_t(u_t) + \sum_{t=1}^{T} \|u_t - u_{t-1}\| + \sum_{t=1}^{T} \left( 1 - \frac{2}{\alpha} \right) f_t(u_t)
\leq \frac{2}{\alpha} \sum_{t=1}^{T} f_t(u_t) + \sum_{t=1}^{T} (f_t(u_t) + \|u_t - u_{t-1}\|)
\]
which implies the naive algorithm is 1-competitive. Otherwise, we have
\[
\sum_{t=1}^{T} (f_t(x_t) + \|x_t - x_{t-1}\|)
\leq \frac{2}{\alpha} \sum_{t=1}^{T} f_t(u_t) + \sum_{t=1}^{T} \|u_t - u_{t-1}\| \leq \frac{2}{\alpha} \sum_{t=1}^{T} (f_t(u_t) + \|u_t - u_{t-1}\|).
\]

We complete the proof by combining (24) and (25).

A.2 Proof of Theorem 2

We will make use of the following basic inequality of squared $\ell_2$-norm [Goel et al., 2019 Lemma 12].
\[
\|x + y\|^2 \leq (1 + \rho)\|x\|^2 + \left( 1 + \frac{1}{\rho} \right)\|y\|^2, \forall \rho > 0.
\]

When $t \geq 2$, we have
\[
f_t(x_t) + \frac{1}{2}\|x_t - x_{t-1}\|^2
\leq f_t(x_t) + \frac{1 + \rho}{2}\|u_t - u_{t-1}\|^2 + \frac{1}{2} \left( 1 + \frac{1}{\rho} \right)\|x_t - x_{t-1} - u_t + u_{t-1}\|^2
\leq f_t(x_t) + \frac{1 + \rho}{2}\|u_t - u_{t-1}\|^2 + \left( 1 + \frac{1}{\rho} \right) (\|u_t - x_t\|^2 + \|u_{t-1} - x_{t-1}\|^2)
\leq f_t(x_t) + \frac{1 + \rho}{2}\|u_t - u_{t-1}\|^2 + \frac{2}{\lambda} \left( 1 + \frac{1}{\rho} \right) (f_t(u_t) - f_t(x_t) + f_{t-1}(u_{t-1}) - f_{t-1}(x_{t-1})).
\]
For $t = 1$, we have
\[
f_1(x_1) + \frac{1}{2}\|x_1 - x_0\|^2
\leq f_1(x_1) + \frac{1 + \rho}{2}\|u_1 - u_0\|^2 + \frac{2}{\lambda} \left( 1 + \frac{1}{\rho} \right) (f_1(u_1) - f_1(x_1)).
\]

Summing over all the iterations, we have
\[
\sum_{t=1}^{T} \left( f_t(x_t) + \frac{1}{2}\|x_t - x_{t-1}\|^2 \right)
\leq \sum_{t=1}^{T} f_t(x_t) + \frac{1 + \rho}{2} \sum_{t=1}^{T} \|u_t - u_{t-1}\|^2 + \frac{2}{\lambda} \left( 1 + \frac{1}{\rho} \right) \sum_{t=1}^{T} (f_t(u_t) - f_t(x_t))
+ \frac{2}{\lambda} \left( 1 + \frac{1}{\rho} \right) \sum_{t=2}^{T} (f_{t-1}(u_{t-1}) - f_{t-1}(x_{t-1}))
\leq \sum_{t=1}^{T} f_t(x_t) + \frac{1 + \rho}{2} \sum_{t=1}^{T} \|u_t - u_{t-1}\|^2 + \frac{4}{\lambda} \left( 1 + \frac{1}{\rho} \right) \sum_{t=1}^{T} (f_t(u_t) - f_t(x_t))
\leq \frac{4}{\lambda} \left( 1 + \frac{1}{\rho} \right) \sum_{t=1}^{T} f_t(u_t) + \frac{1 + \rho}{2} \sum_{t=1}^{T} \|u_t - u_{t-1}\|^2 + \left( 1 - \frac{4}{\lambda} \left( 1 + \frac{1}{\rho} \right) \right) \sum_{t=1}^{T} f_t(x_t).
\]
First, we consider the case that
\[ 1 - \frac{4}{\lambda} \left( 1 + \frac{1}{\rho} \right) \leq 0 \iff \frac{\lambda}{4} \leq 1 + \frac{1}{\rho} \]  
and have
\[
\sum_{t=1}^{T} \left( f_t(x_t) + \frac{1}{2} \| x_t - x_{t-1} \|^2 \right) \leq \sum_{t=1}^{T} \left( f_t(u_t) + \frac{1}{2} \| u_t - u_{t-1} \|^2 \right) 
\]
(27), (28)
\[
\leq \frac{4}{\lambda} \left( 1 + \frac{1}{\rho} \right) \sum_{t=1}^{T} \sum_{t=1}^{T} \frac{1 + \rho}{2} \sum_{t=1}^{T} \| u_t - u_{t-1} \|^2 
\]
(27), (28)
\[
\leq \max \left( \frac{4}{\lambda} \left( 1 + \frac{1}{\rho} \right), 1 + \rho \right) \sum_{t=1}^{T} \left( f_t(u_t) + \frac{1}{2} \| u_t - u_{t-1} \|^2 \right).
\]
To minimize the competitive ratio, we set
\[
\frac{4}{\lambda} \left( 1 + \frac{1}{\rho} \right) = 1 + \rho \Rightarrow \rho = \frac{4}{\lambda}
\]
and obtain
\[
\sum_{t=1}^{T} \left( f_t(x_t) + \frac{1}{2} \| x_t - x_{t-1} \|^2 \right) \leq \left( 1 + \frac{4}{\lambda} \right) \sum_{t=1}^{T} \left( f_t(u_t) + \frac{1}{2} \| u_t - u_{t-1} \|^2 \right). \tag{29}
\]
Next, we study the case that
\[
1 - \frac{4}{\lambda} \left( 1 + \frac{1}{\rho} \right) \geq 0 \iff \frac{\lambda}{4} \geq 1 + \frac{1}{\rho}
\]
which only happens when \( \lambda > 4 \). Then, we have
\[
\sum_{t=1}^{T} \left( f_t(x_t) + \frac{1}{2} \| x_t - x_{t-1} \|^2 \right) \leq \sum_{t=1}^{T} \left( f_t(u_t) + \frac{1}{2} \| u_t - u_{t-1} \|^2 \right)
\]
(27), (28)
\[
\sum_{t=1}^{T} \left( f_t(x_t) + \frac{1}{2} \| x_t - x_{t-1} \|^2 \right) \leq \sum_{t=1}^{T} \left( f_t(u_t) + \frac{1}{2} \| u_t - u_{t-1} \|^2 \right)
\]
which is worse than (29). So, we keep (29) as the final result.

**A.3 Proof of Theorem 3**

Since \( f_t(\cdot) \) is convex, the objective function of (10) is \( \gamma \)-strongly convex. From the quadratic growth property of strongly convex functions (Hazan and Kale 2011), we have
\[
f_t(x_t) + \frac{\gamma}{2} \| x_t - x_{t-1} \|^2 + \frac{\gamma}{2} \| u - x_t \|^2 \leq f_t(u) + \frac{\gamma}{2} \| u - x_{t-1} \|^2, \quad \forall u \in \mathcal{X}. \tag{30}
\]

Similar to previous studies (Bansal et al. 2015), the analysis uses an amortized local competitiveness argument, using the potential function \( c \| x_t - u_t \|^2 \). We proceed to bound \( f_t(x_t) + \frac{1}{2} \| x_t - x_{t-1} \|^2 + c \| x_t - u_t \|^2 - c \| x_{t-1} - u_{t-1} \|^2 \), and have
\[
f_t(x_t) + \frac{1}{2} \| x_t - x_{t-1} \|^2 + c \| x_t - u_t \|^2 - c \| x_{t-1} - u_{t-1} \|^2
\]
(29)
\[
\leq f_t(x_t) + \frac{1}{2} \| x_t - x_{t-1} \|^2 + c \| x_t - v_t \|^2 + 2 \| v_t - u_t \|^2 - c \| x_{t-1} - u_{t-1} \|^2
\]
(29)
\[
\leq \left(1 + \frac{4c}{\lambda} \right) f_t(x_t) + \frac{1}{2} \| x_t - x_{t-1} \|^2 + \frac{4c}{\lambda} f_t(u_t) - c \| x_{t-1} - u_{t-1} \|^2
\]
(29)
\[
= \left(1 + \frac{4c}{\lambda} \right) \left( f_t(x_t) + \frac{\lambda}{2(\lambda + 4c)} \| x_t - x_{t-1} \|^2 \right) + \frac{4c}{\lambda} f_t(u_t) - c \| x_{t-1} - u_{t-1} \|^2.
\]
Suppose
\[ \frac{\lambda}{\lambda + 4c} \leq \gamma, \]  
we have
\[ f_t(x_t) + \frac{1}{2}||x_t - x_{t-1}||^2 + c||x_t - u_t||^2 - c||x_{t-1} - u_{t-1}||^2 \]
\[ \leq \left(1 + \frac{4c}{\lambda}\right) \left( f_t(x_t) + \frac{\gamma}{2}||x_t - x_{t-1}||^2 \right) + \frac{4c}{\lambda} f_t(u_t) - c||x_t - u_{t-1}||^2 \]
\[ \leq \left(1 + \frac{4c}{\lambda}\right) \left( f_t(x_t) + \frac{\gamma}{2}||x_t - x_{t-1}||^2 - \frac{\gamma}{2}||x_t - x_{t-1}||^2 \right) + \frac{4c}{\lambda} f_t(u_t) - c||x_t - u_{t-1}||^2 \]
\[ = \left(1 + \frac{8c}{\lambda}\right) f_t(u_t) + \frac{\gamma (\lambda + 4c)}{2\lambda} ||x_t - x_{t-1}||^2 - \frac{\gamma (\lambda + 4c)}{2\lambda} ||x_t - x_{t-1}||^2 - c||x_t - u_{t-1}||^2. \]

Summing over all the iterations and assuming \( x_0 = u_0 \), we have
\[ \sum_{t=1}^{T} \left( f_t(x_t) + \frac{1}{2}||x_t - x_{t-1}||^2 \right) + c||x_T - u_T||^2 \]
\[ \leq \left(1 + \frac{8c}{\lambda}\right) \sum_{t=1}^{T} f_t(u_t) + \frac{\gamma (\lambda + 4c)}{2\lambda} \sum_{t=1}^{T} ||x_t - x_{t-1}||^2 \]
\[ - \frac{\gamma (\lambda + 4c)}{2\lambda} \sum_{t=1}^{T} ||x_t - x_{t-1}||^2 - c \sum_{t=1}^{T} ||x_t - u_{t-1}||^2 \]
\[ \leq \left(1 + \frac{8c}{\lambda}\right) \sum_{t=1}^{T} f_t(u_t) + \frac{\gamma (\lambda + 4c)}{2\lambda} \sum_{t=1}^{T} ||x_t - x_{t-1}||^2 - \left( \frac{\gamma (\lambda + 4c)}{2\lambda} + c \right) \sum_{t=1}^{T} ||x_t - u_{t-1}||^2 \]
\[ \leq \left(1 + \frac{8c}{\lambda}\right) \sum_{t=1}^{T} f_t(u_t) + \frac{\gamma (\lambda + 4c)}{2\lambda} \sum_{t=1}^{T} ||x_t - x_{t-1}||^2 - \left( \frac{\gamma (\lambda + 4c)}{2\lambda} + c \right) \sum_{t=1}^{T} \left( f_t(u_t) + \frac{1}{2} ||x_t - u_{t-1}||^2 \right) \]
\[ \leq \max \left(1 + \frac{8c}{\lambda}, \left( \frac{\gamma (\lambda + 4c)}{2\lambda} + c \right) \frac{2}{\rho} \right) \sum_{t=1}^{T} \left( f_t(u_t) + \frac{1}{2} ||x_t - u_{t-1}||^2 \right) \]
where in the penultimate inequality we assume
\[ \frac{\gamma (\lambda + 4c)}{2\lambda} \leq \left( \frac{\gamma (\lambda + 4c)}{2\lambda} + c \right) \frac{1}{1 + \rho} \Leftrightarrow \frac{\gamma (\lambda + 4c)}{2\lambda} \leq \frac{c}{\rho}. \]  

Next, we minimize the competitive ratio under the constraints in (31) and (32), which can be summarized as
\[ \frac{\lambda}{\lambda + 4c} \leq \gamma \leq \frac{\lambda}{\lambda + 4c} \frac{2c}{\rho}. \]

We first set \( c = \frac{\rho}{4} \) and \( \gamma = \frac{\lambda}{\lambda + 4c} \), and obtain
\[ \sum_{t=1}^{T} \left( f_t(x_t) + \frac{1}{2} ||x_t - x_{t-1}||^2 \right) \leq \max \left(1 + \frac{4\rho}{\lambda}, 1 + \frac{1}{\rho} \right) \sum_{t=1}^{T} \left( f_t(u_t) + \frac{1}{2} ||x_t - u_{t-1}||^2 \right). \]

Then, we set
\[ 1 + \frac{4\rho}{\lambda} = 1 + \frac{1}{\rho} \Rightarrow \rho = \frac{\sqrt{\lambda}}{2}. \]
As a result, the competitive ratio is
\[ 1 + \frac{1}{\rho} = 1 + \frac{2}{\lambda}, \]
and the parameter is
\[ \gamma = \frac{\lambda}{\lambda + 4c} = \frac{\lambda}{\lambda + 2\rho} = \frac{\lambda}{\lambda + \sqrt{\lambda}}. \]

### A.4 Proof of Theorem 4

The analysis is similar to the proof of Theorem 3 of Zhang et al. [2018a]. In the analysis, we need to specify the behavior of the meta-algorithm and expert-algorithm at \( t = 0 \). To simplify the presentation, we set
\[ x_0 = 0, \quad \text{and} \quad x_0^\eta = 0, \quad \forall \eta \in \mathcal{H}. \]
\[
(33)
\]

First, we bound the dynamic regret with switching cost of the meta-algorithm w.r.t. all experts simultaneously.

**Lemma 1** Under Assumptions 2 and 3, and setting \( \beta = \frac{2}{(2G + 1)D} \sqrt{\frac{2}{5T}} \), we have
\[
\sum_{t=1}^{T} \left( s_t(x_t) + \|x_t - x_{t-1}\| \right) - \sum_{t=1}^{T} \left( s_t(x^\eta_t) + \|x^\eta_t - x^\eta_{t-1}\| \right) \leq (2G + 1)D \sqrt{\frac{5T}{s}} \left( \ln \frac{1}{w_1} + 1 \right) \quad (34)
\]
for each \( \eta \in \mathcal{H} \).

Next, we bound the dynamic regret with switching cost of each expert w.r.t. any comparator sequence \( u_0, u_1, \ldots, u_T \in \mathcal{X} \).

**Lemma 2** Under Assumptions 2 and 3, we have
\[
\sum_{t=1}^{T} \left( s_t(x^\eta_t) + \|x^\eta_t - x^\eta_{t-1}\| \right) - \sum_{t=1}^{T} s_t(u_t) \leq \frac{D^2}{2\eta} + \frac{D}{\eta} \sum_{t=1}^{T} \|u_t - u_{t-1}\| + \eta T \left( \frac{G^2}{2} + G \right). \quad (35)
\]

Then, we show that for any sequence of comparators \( u_0, u_1, \ldots, u_T \in \mathcal{X} \) there exists an \( \eta_k \in \mathcal{H} \) such that the R.H.S. of (35) is almost minimal. If we minimize the R.H.S. of (35) exactly, the optimal step size is
\[
\eta^*(P_T) = \sqrt{\frac{D^2 + 2DP_T}{T(G^2 + 2G)}}, \quad (36)
\]

From Assumption 3, we have the following bound of the path-length
\[
0 \leq P_T = \sum_{t=1}^{T} \|u_t - u_{t-1}\| \leq TD. \quad (37)
\]
Thus
\[
\sqrt{\frac{D^2}{T(G^2 + 2G)}} \leq \eta^*(P_T) \leq \sqrt{\frac{D^2 + 2TD^2}{T(G^2 + 2G)}}.
\]

From our construction of \( \mathcal{H} \) in (17), it is easy to verify that
\[
\min \mathcal{H} = \sqrt{\frac{D^2}{T(G^2 + 2G)}}, \quad \text{and} \quad \max \mathcal{H} \geq \sqrt{\frac{D^2 + 2TD^2}{T(G^2 + 2G)}}.
\]

As a result, for any possible value of \( P_T \), there exists a step size \( \eta_k \in \mathcal{H} \) with \( k \) defined in (19), such that
\[
\eta_k = 2^{k-1} \sqrt{\frac{D^2}{T(G^2 + 2G)}} \leq \eta^*(P_T) \leq 2 \eta_k. \quad (38)
\]
Plugging $\eta_k$ into (35), the dynamic regret with switching cost of expert $E^{n_k}$ is given by
\[
\sum_{t=1}^{T} \left( s_t(x_t^{n_k}) + \|x_t^{n_k} - x_{t-1}^{n_k}\| \right) - \sum_{t=1}^{T} s_t(u_t) \\
\leq \frac{D^2}{2\eta_k} + \frac{D}{\eta_k} \sum_{t=1}^{T} \|u_t - u_{t-1}\| + \eta_k T \left( \frac{G^2}{2} + G \right) \\
\leq \frac{D^2}{\eta^*(P_T)} + \frac{2D}{\eta^*(P_T)} \sum_{t=1}^{T} \|u_t - u_{t-1}\| + \eta^*(P_T) T \left( \frac{G^2}{2} + G \right) \\
\leq \frac{3}{2} \sqrt{T(G^2 + 2G)(D^2 + 2DP_T)}.
\]

From (13), we know the initial weight of expert $E^{n_k}$ is
\[
w_0^{n_k} = \frac{C}{k(k+1)} \geq \frac{1}{k(k+1)^2}.
\]
Combining with (34), we obtain the relative performance of the meta-algorithm w.r.t. expert $E^{n_k}$:
\[
\sum_{t=1}^{T} \left( s_t(x_t) + \|x_t - x_{t-1}\| \right) - \sum_{t=1}^{T} s_t(x_t^{n_k}) + \|x_t^{n_k} - x_{t-1}^{n_k}\| \leq (2G + 1)D \sqrt{\frac{5T}{8}} [1 + 2 \ln(k + 1)].
\]

From (39) and (40), we derive the following upper bound for dynamic regret with switching cost
\[
\sum_{t=1}^{T} \left( s_t(x_t) + \|x_t - x_{t-1}\| \right) - \sum_{t=1}^{T} s_t(u_t) \\
\leq \frac{3}{2} \sqrt{T(G^2 + 2G)(D^2 + 2DP_T)} + (2G + 1)D \sqrt{\frac{5T}{8}} [1 + 2 \ln(k + 1)].
\]

Finally, from Assumption 1 we have
\[
f_t(x_t) - f_t(u_t) \leq \langle \nabla f_t(x_t), x_t - u_t \rangle \leq s_t(x_t) - s_t(u_t).
\]
We complete the proof by combining (41) and (42).

### A.5 Proof of Theorem 5

The analysis is similar to that of Theorem 4. The difference is that we need to take into account the lookahead property of the meta-algorithm and the expert-algorithm.

First, we bound the dynamic regret with switching cost of the meta-algorithm w.r.t. all experts simultaneously.

**Lemma 3** Under Assumption 3 and setting $\beta = \frac{1}{T} \sqrt{2}$, we have
\[
\sum_{t=1}^{T} \left( s_t(x_t) + \|x_t - x_{t-1}\| \right) - \sum_{t=1}^{T} s_t(x_t^{n_k}) + \|x_t^{n_k} - x_{t-1}^{n_k}\| \leq D \sqrt{\frac{T}{2}} \left( \frac{1}{u_0^{n_k}} + 1 \right)
\]
for each $\eta \in \mathcal{H}$.

Combining Lemma 3 with Assumption 1 we have
\[
\sum_{t=1}^{T} \left( f_t(x_t) + \|x_t - x_{t-1}\| \right) - \sum_{t=1}^{T} f_t(x_t^{n_k}) + \|x_t^{n_k} - x_{t-1}^{n_k}\| \leq D \sqrt{\frac{T}{2}} \left( \frac{1}{u_0^{n_k}} + 1 \right)
\]
for each $\eta \in \mathcal{H}$.

Next, we bound the dynamic regret with switching cost of each expert w.r.t. any comparator sequence $u_0, u_1, \ldots, u_T \in X$.  

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We complete the proof by summing (48) and (49) together.

As a result, for any possible value of \( P \) the proof is built upon a lower bound of competitive ratio \cite{Argue et al., 2020a}. By setting \( \gamma \) in Lemma 12 of \cite{Argue et al., 2020a}, we can guarantee that Assumption 3 is satisfied. Then, we choose \( \mu \) in Lemma 4.

The rest of the proof is almost identical to that of Theorem 4. We will show that for any sequence of comparators \( u_0, u_1, \ldots, u_T \in \mathcal{X} \) there exists an \( \eta_k \in \mathcal{H} \) such that the R.H.S. of (43) is almost minimal. If we minimize the R.H.S. of (45) exactly, the optimal step size is

\[
\eta^*(P_T) = \frac{\sqrt{D^2 + 2DP_T}}{T}.
\] (46)

From (37), we know that

\[
\sqrt{\frac{D^2}{T}} \leq \eta^*(P_T) \leq \sqrt{\frac{D^2 + 2TD^2}{T}}.
\]

From our construction of \( \mathcal{H} \) in (22), it is easy to verify that

\[
\min \mathcal{H} = \sqrt{\frac{D^2}{T}}, \text{ and } \max \mathcal{H} \geq \sqrt{\frac{D^2 + 2TD^2}{T}}.
\]

As a result, for any possible value of \( P_T \), there exists a step size \( \eta_k \in \mathcal{H} \) with \( k \) defined in (19), such that

\[
\eta_k = 2^{k-1} \sqrt{\frac{D^2}{T}} \leq \eta^*(P_T) \leq 2\eta_k.
\] (47)

Plugging \( \eta_k \) into (45), the dynamic regret with switching cost of expert \( E^{\eta_k} \) is given by

\[
\sum_{t=1}^{T} \left( f_t(x_t^{\eta_k}) + \| x_t^{\eta_k} - x_{t-1}^{\eta_k} \| \right) - \sum_{t=1}^{T} f_t(u_t)
\leq \frac{D^2}{2\eta_k} + \frac{D}{\eta_k} \sum_{t=1}^{T} \| u_t - u_{t-1} \| + \frac{\eta_k T}{2}
\leq \frac{D^2}{\eta^*(P_T)} + \frac{2D}{\eta^*(P_T)} \sum_{t=1}^{T} \| u_t - u_{t-1} \| + \frac{\eta^*(P_T)T}{2}
\leq \frac{3}{2} \sqrt{T(D^2 + 2DP_T)}.
\] (48)

From Step 2 of Algorithm 3, we know the initial weight of expert \( E^{\eta_k} \) is

\[
w_0^{\eta_k} = \frac{C}{k(k+1)} \geq \frac{1}{k(k+1)} \geq \frac{1}{(k+1)^2}.
\]

Combining with (43), we obtain the relative performance of the meta-algorithm w.r.t. expert \( E^{\eta_k} \):

\[
\sum_{t=1}^{T} \left( f_t(x_t) + \| x_t - x_{t-1} \| \right) - \sum_{t=1}^{T} \left( f_t(x_t^{\eta_k}) + \| x_t^{\eta_k} - x_{t-1}^{\eta_k} \| \right) \leq D \sqrt{T \left[ 1 + 2 \ln(k+1) \right]}.
\] (49)

We complete the proof by summing (48) and (49) together.

**A.6 Proof of Theorem 6**

The proof is built upon a lower bound of competitive ratio \cite{Argue et al., 2020a}. By setting \( \gamma = \frac{D}{2\sqrt{d}} \) in Lemma 12 of \cite{Argue et al., 2020a}, we can guarantee that Assumption 3 is satisfied. Then, we choose \( \mu = 0, \lambda = 1/\gamma \) in that lemma, and obtain the conclusion below.

**Lemma 5** For any online algorithm \( A \) and any fixed value of \( d \), there exists a sequence of convex functions \( f_1(\cdot), \ldots, f_d(\cdot) \) over the domain \([-\frac{D}{2\sqrt{d}}, \frac{D}{2\sqrt{d}}]^d \) in the lookahead setting such that
We consider two cases: \( \tau < D \) and \( \tau \geq D \). When \( \tau < D \), from Lemma 5 with \( d = T \), we know that the dynamic regret with switching cost w.r.t. a fixed point \( u \) is at least \( \Omega(D\sqrt{T}) \).

Next, we consider the case \( \tau \geq D \). Without loss of generality, we assume \( \lfloor \tau/D \rfloor \) divides \( T \). Then, we partition \( T \) into \( \lfloor \tau/D \rfloor \) successive stages, each of which contains \( T/\lfloor \tau/D \rfloor \) rounds. Applying Lemma 5 to each stage, we conclude that there exists a sequence of convex functions \( f_1(\cdot), \ldots, f_T(\cdot) \) over the domain \([-\frac{D}{2\sqrt{d}}, \frac{D}{2\sqrt{d}}]^d\) where \( d = T/\lfloor \tau/D \rfloor \) in the lookahead setting such that

1. the sum of the hitting cost and the switching cost of any online algorithm is at least \( \frac{3\gamma d}{4} = \frac{3D\sqrt{D}}{8} \);
2. there exist a fixed point \( u \) whose hitting cost is 0.

Thus, the dynamic regret with switching cost w.r.t. a fixed point \( u \) is at least \( \frac{3D}{8} \sqrt{T} \lfloor \tau/D \rfloor = \Omega(\sqrt{TD}\tau) \).

We complete the proof by combining the results of the above two cases.

### B Proof of supporting lemmas

We provide the proof of all the supporting lemmas.

#### B.1 Proof of Lemma 1

Based on the prediction rule of the meta-algorithm, we upper bound the switching cost when \( t \geq 2 \) as follows:

\[
\|x_t - x_{t-1}\| = \sum_{\eta \in \mathcal{H}} w^\eta_t x^\eta_t - \sum_{\eta \in \mathcal{H}} w^\eta_{t-1} x^\eta_{t-1} = \sum_{\eta \in \mathcal{H}} w^\eta_t (x^\eta_t - x) - \sum_{\eta \in \mathcal{H}} w^\eta_{t-1} (x^\eta_{t-1} - x) \\
\leq \sum_{\eta \in \mathcal{H}} w^\eta_t (x^\eta_t - x) - \sum_{\eta \in \mathcal{H}} w^\eta_{t-1} (x^\eta_{t-1} - x) + \sum_{\eta \in \mathcal{H}} w^\eta_t (x^\eta_{t-1} - x) - \sum_{\eta \in \mathcal{H}} w^\eta_{t-1} (x^\eta_{t-1} - x) \\
= \sum_{\eta \in \mathcal{H}} w^\eta_t (x^\eta_t - x^\eta_{t-1}) + \sum_{\eta \in \mathcal{H}} (w^\eta_t - w^\eta_{t-1}) (x^\eta_{t-1} - x) \\
\leq \sum_{\eta \in \mathcal{H}} w^\eta_t \|x^\eta_t - x^\eta_{t-1}\| + \sum_{\eta \in \mathcal{H}} |w^\eta_t - w^\eta_{t-1}| \|x^\eta_{t-1} - x\| \\
\leq \sum_{\eta \in \mathcal{H}} w^\eta_t \|x^\eta_t - x^\eta_{t-1}\| + D \sum_{\eta \in \mathcal{H}} |w^\eta_t - w^\eta_{t-1}| = \sum_{\eta \in \mathcal{H}} w^\eta_t \|x^\eta_t - x^\eta_{t-1}\| + D \|w_t - w_{t-1}\|, \\
\tag{50}
\end{align*}

where \( x \) is an arbitrary point in \( \mathcal{X} \), and \( w_t = (w^\eta_t)_{\eta \in \mathcal{H}} \in \mathbb{R}^N \). When \( t = 1 \), from (33), we have

\[
\|x_1 - x_0\| = \|x_1\| = \sum_{\eta \in \mathcal{H}} w^\eta_1 x^\eta_1 = \sum_{\eta \in \mathcal{H}} w^\eta_1 \|x^\eta_1\| = \sum_{\eta \in \mathcal{H}} w^\eta_1 \|x^\eta_1 - x_0^\eta\|. \tag{51}
\]

\[
\]
Then, the relative loss of the meta-algorithm w.r.t. expert $E^\eta$ can be decomposed as

$$
\sum_{t=1}^{T} \left( \sum_{\eta,\exists H} w^\eta_t \|x^\eta_t - x_{t-1}\| \right) - \sum_{t=1}^{T} \left( \sum_{\eta,\exists H} w^\eta_t \|x^\eta_t - x_{t-1}\| \right) + D \sum_{t=2}^{T} \|w_t - w_{t-1}\|_1 
$$

We proceed to bound $A$ and $\|w_t - w_{t-1}\|_1$ in (52). Notice that $A$ is the regret of the meta-algorithm w.r.t. expert $E^\eta$. From Assumptions 2 and 3, we have

$$
\|\nabla f_t(x_t), x^\eta_t - x_t\| \leq \|\nabla f_t(x_t), x^\eta_t - x_t\| \leq GD.
$$

Thus, we have

$$
-GD \leq \ell_t(x^\eta_t) \leq (G + 1)D, \forall \eta \in \mathcal{H}.
$$

According to the standard analysis of Hedge [Zhang et al., 2018a, Lemma 1] and (53), we have

$$
\sum_{t=1}^{T} \left( \sum_{\eta,\exists H} w^\eta_t \ell_t(x^\eta_t) - \ell_t(x_t) \right) \leq \frac{1}{\beta} \ln \frac{1}{w^\eta_1} + \frac{\beta T (2G + 1)^2 D^2}{8}.
$$

Next, we bound $\|w_t - w_{t-1}\|_1$, which measures the stability of the meta-algorithm, i.e., the change of coefficients between successive rounds. Because the Hedge algorithm is translation invariant, we can subtract $D/2$ from $\ell_t(x^\eta_t)$ such that

$$
|\ell_t(x^\eta_t) - D/2| \leq (G + 1/2)D, \forall \eta \in \mathcal{H}.
$$

It is well-known that Hedge can be treated as a special case of “Follow-the-Regularized-Leader” with entropic regularization [Shalev-Shwartz, 2011]

$$
R(w) = \sum_i w_i \log w_i
$$

over the probability simplex, and $R(\cdot)$ is 1-strongly convex w.r.t. the $\ell_1$-norm. In other words, we have

$$
w_{t+1} = \arg\min_{w \in \Delta} \left\{ -\frac{1}{\beta} \log(w_1) + \sum_{i=1}^{t} g_i, w \right\} + \frac{1}{\beta} R(w), \forall t \geq 1
$$

where $\Delta \subseteq \mathbb{R}^N$ is the probability simplex, and $g_t = [\ell_t(x^\eta_t) - D/2]_{\eta \in \mathcal{H}} \subseteq \mathbb{R}^N$. From the stability property of Follow-the-Regularized-Leader [Duchi et al., 2012, Lemma 2], we have

$$
\|w_t - w_{t-1}\|_1 \leq \beta \|g_{t-1}\|_\infty \leq \beta (G + 1/2)D, \forall t \geq 2.
$$

Then

$$
\sum_{t=2}^{T} \|w_t - w_{t-1}\|_1 \leq \frac{\beta (T-1)(2G + 1)D}{2}.
$$

Substituting (54) and (56) into (52), we have

$$
\sum_{t=1}^{T} \left( s_t(x_t) + \|x_t - x_{t-1}\| \right) - \sum_{t=1}^{T} \left( s_t(x^\eta_t) + \|x^\eta_t - x_{t-1}\| \right) \leq \frac{1}{\beta} \ln \frac{1}{w^\eta_1} + \frac{\beta T (2G + 1)^2 D^2}{8} + \frac{\beta (T-1)(2G + 1)D^2}{2} \leq \frac{1}{\beta} \ln \frac{1}{w^\eta_1} + \frac{5\beta T (2G + 1)^2 D^2}{8}.
$$

We complete the proof by setting $\beta = \frac{2}{(2G+1)D} \sqrt{\frac{8}{2T}}$. 

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B.2 Proof of Lemma 2

First, we bound the dynamic regret of the expert-algorithm. Define
\[ x_{t+1}^\eta = x_t^\eta - \eta \nabla f_t(x_t). \]

Following the analysis of Ader [Zhang et al., 2018a, Theorems 1 and 6], we have
\[ s_t(x_t^\eta) - s_t(u_t) \leq \frac{1}{2\eta} \| x_t^\eta - u_t \|^2 + D \eta \sum_{t=1}^T \| u_{t+1} - u_t \| + \frac{\eta T}{2} G^2. \] (57)

Summing the above inequality over all iterations, we have
\[ \sum_{t=1}^T (s_t(x_t^\eta) - s_t(u_t)) \leq \frac{1}{2\eta} \| x_T^\eta - u_1 \|^2 + D \eta \sum_{t=1}^T \| u_{t+1} - u_t \| + \frac{\eta T}{2} G^2. \] (58)

Since (57) holds when \( u_{T+1} = u_T \), we have
\[ \sum_{t=1}^T (s_t(x_t^\eta) - s_t(u_t)) \leq \frac{1}{2\eta} D^2 + D \eta \sum_{t=1}^T \| u_{t+1} - u_t \| + \frac{\eta T}{2} G^2. \] (59)

Next, we bound the switching cost of the expert-algorithm. To this end, we have
\[ \sum_{t=1}^T \| x_t^\eta - x_{t-1}^\eta \| = \sum_{t=0}^{T-1} \| x_{t+1}^\eta - x_t^\eta \| \leq \sum_{t=0}^{T-1} \| \bar{x}_{t+1}^\eta - x_t^\eta \| \leq D \sum_{t=0}^{T-1} \| \eta \nabla f_t(x_t) \| \leq \eta T G. \] (60)

We complete the proof by combining (58) with (59).

B.3 Proof of Lemma 3

We reuse the first part of the proof of Lemma 1 and start from (52). To bound \( A \), we need to analyze the behavior of the lookahead Hedge. To this end, we prove the following lemma.

Lemma 6 The meta-algorithm in Algorithm 3 satisfies
\[ \sum_{t=1}^T \left( \sum_{\eta \in \mathcal{H}} w_t^\eta \ell_t(x_t^\eta) - \ell_t(x_t^\eta) \right) \leq \frac{1}{\beta} \ln \frac{1}{w_0^\eta} - \frac{1}{2\beta} \sum_{t=1}^T \| w_t - w_{t-1} \|^2 \] (61)
for any \( \eta \in \mathcal{H} \).
Substituting (60) into (52), we have
\[
\sum_{t=1}^{T} \left( s_t(x_t) + \|x_t - x_{t-1}\| \right) - \sum_{t=1}^{T} \left( s_t(x^0_t) + \|x^0_t - x^0_{t-1}\| \right) \\
\leq \frac{1}{\beta} \ln \frac{1}{w_0^T} - \frac{1}{2\beta} \sum_{t=1}^{T} \|w_t - w_{t-1}\|^2_1 + D \sum_{t=2}^{T} \|w_t - w_{t-1}\|_1 \\
\leq \frac{1}{\beta} \ln \frac{1}{w_0^T} - \frac{1}{2\beta} \sum_{t=1}^{T} \|w_t - w_{t-1}\|^2_1 + \sum_{t=2}^{T} \left( \frac{1}{2\beta} \|w_t - w_{t-1}\|^2_1 + \frac{\beta D^2}{2} \right) \\
\leq \frac{1}{\beta} \ln \frac{1}{w_0^T} + \frac{\beta TD^2}{2} = D \sqrt{\frac{T}{2} \left( \ln \frac{1}{w_0^T} + 1 \right)}
\]  

where we set \( \beta = \frac{1}{D \sqrt{T}} \).

**B.4 Proof of Lemma 6**

To simplify the notation, we define
\[
W_0 = \sum_{\eta \in \mathcal{H}} w_0^\eta = 1, \quad L^\eta_t = \sum_{i=1}^{t} \ell_i(x^\eta_i), \quad \text{and} \quad W_t = \sum_{\eta \in \mathcal{H}} w_0^\eta e^{-\beta L^\eta_t}, \quad \forall t \geq 1.
\]

From the updating rule in (20), it is easy to verify that
\[
w_t^\eta = \frac{w_0^\eta e^{-\beta L^\eta_t}}{W_t}, \quad \forall t \geq 1.
\]  

First, we have
\[
\ln W_T = \ln \left( \sum_{\eta \in \mathcal{H}} w_0^\eta e^{-\beta L^\eta_T} \right) \geq \ln \left( \max_{\eta \in \mathcal{H}} w_0^\eta e^{-\beta L^\eta_T} \right) = -\beta \min_{\eta \in \mathcal{H}} \left( L^\eta_T + \frac{1}{\beta} \ln \frac{1}{w_0^\eta} \right).
\]  

Next, we bound the related quantity \( \ln(W_t/W_{t-1}) \) as follows. For any \( \eta \in \mathcal{H} \), we have
\[
\ln \left( \frac{W_t}{W_{t-1}} \right) \geq \ln \left( \frac{w_0^\eta e^{-\beta L^\eta_t}}{w_0^\eta e^{-\beta L^\eta_{t-1}}} \right) = \ln \left( \frac{w_{t-1}^\eta}{w_t^\eta} \right) - \beta \ell_t(x^\eta_t).
\]  

Then, we have
\[
\ln \left( \frac{W_t}{W_{t-1}} \right) = \ln \left( \frac{W_t}{W_{t-1}} \right) \sum_{\eta \in \mathcal{H}} w_t^\eta = \sum_{\eta \in \mathcal{H}} w_t^\eta \ln \left( \frac{W_t}{W_{t-1}} \right) \\
64 \sum_{\eta \in \mathcal{H}} w_t^\eta \ln \left( \frac{w_{t-1}^\eta}{w_t^\eta} \right) - \beta \sum_{\eta \in \mathcal{H}} w_t^\eta \ell_t(x^\eta_t) \leq -\frac{1}{2} \|w_t - w_{t-1}\|^2_1 - \beta \sum_{\eta \in \mathcal{H}} w_t^\eta \ell_t(x^\eta_t)
\]  

where the last inequality is due to Pinsker’s inequality [Cover and Thomas [2006] Lemma 11.6.1]. Thus
\[
\ln W_T = \ln W_0 + \sum_{t=1}^{T} \ln \left( \frac{W_t}{W_{t-1}} \right) \geq \sum_{t=1}^{T} \left( -\frac{1}{2} \|w_t - w_{t-1}\|^2_1 - \beta \sum_{\eta \in \mathcal{H}} w_t^\eta \ell_t(x^\eta_t) \right).
\]  

Combining (63) with (66), we obtain
\[
-\beta \min_{\eta \in \mathcal{H}} \left( L^\eta_T + \frac{1}{\beta} \ln \frac{1}{w_0^T} \right) \leq \sum_{t=1}^{T} \left( -\frac{1}{2} \|w_t - w_{t-1}\|^2_1 - \beta \sum_{\eta \in \mathcal{H}} w_t^\eta \ell_t(x^\eta_t) \right)
\]

We complete the proof by rearranging the above inequality.
B.5 Proof of Lemma 4

The analysis is similar to that of Theorem 10 of [Chen et al., 2018], which relies on a strong condition

$$x^*_\eta = x^*_{t-1} - \eta \nabla f_t(x^*_\eta).$$

Note that the above equation is essentially the vanishing gradient condition of $x^*_\eta$ when (21) is unconstrained. In contrast, we only make use of the first-order optimality criterion of $x^*_\eta$ [Boyd and Vandenberghe, 2004], i.e.,

$$\langle \nabla f_t(x^*_\eta) + \frac{1}{\eta} (x^*_\eta - x^*_{t-1}), y - x^*_\eta \rangle \geq 0, \ \forall y \in \mathcal{X}$$  \hspace{1cm} (67)

which is much weaker.

From the convexity of $f_t(\cdot)$, we have

$$f_t(x^*_\eta) - f_t(u_t) \leq \langle \nabla f_t(x^*_\eta), x^*_\eta - u_t \rangle$$

$$\leq \frac{1}{\eta} \langle x^*_\eta - x^*_{t-1}, u_t - x^*_\eta \rangle = \frac{1}{2\eta} \left( \|x^*_\eta - u_t\|^2 - \|x^*_\eta - x^*_{t-1}\|^2 - \|x^*_{t-1} - u_t\|^2 + \|x^*_\eta - x^*_{t-1}\|^2 \right)$$

$$\leq \frac{1}{2\eta} \left( \|x^*_\eta - x^*_{t-1}\|^2 - \|x^*_\eta - u_t\|^2 \right) + \frac{D}{\eta} \|u_t - u_{t-1}\| - \frac{1}{2\eta} \|x^*_\eta - x^*_{t-1}\|^2.$$

Summing the above inequality over all iterations, we have

$$\sum_{t=1}^{T} (f_t(x^*_\eta) - f_t(u_t)) \leq \frac{1}{2\eta} \|x^*_0 - u_0\|^2 + \frac{D}{\eta} \sum_{t=1}^{T} \|u_t - u_{t-1}\| - \frac{1}{2\eta} \sum_{t=1}^{T} \|x^*_\eta - x^*_{t-1}\|^2$$

\hspace{1cm} \leq \frac{1}{2\eta} D^2 + \frac{D}{\eta} \sum_{t=1}^{T} \|u_t - u_{t-1}\| - \frac{1}{2\eta} \sum_{t=1}^{T} \|x^*_\eta - x^*_{t-1}\|^2.$$  \hspace{1cm} (68)

Then, the dynamic regret with switching cost can be upper bounded as follows

$$\sum_{t=1}^{T} (f_t(x^*_\eta) - \|x^*_\eta - x^*_{t-1}\| - f_t(u_t))$$

$$\leq \frac{1}{2\eta} D^2 + \frac{D}{\eta} \sum_{t=1}^{T} \|u_t - u_{t-1}\| - \frac{1}{2\eta} \sum_{t=1}^{T} \|x^*_\eta - x^*_{t-1}\|^2 + \frac{1}{2\eta} \sum_{t=1}^{T} \|x^*_\eta - x^*_{t-1}\|^2 + \frac{\eta}{2} \sum_{t=1}^{T} \|x^*_\eta - x^*_{t-1}\|^2$$

$$\leq \frac{1}{2\eta} D^2 + \frac{D}{\eta} \sum_{t=1}^{T} \|u_t - u_{t-1}\| + \frac{\eta T}{2}.$$